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D-BRANES IN NONCOMMUTATIVE FIELD THEORY[*] [†]

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Abstract

A mathematical introduction to the classical solutions of noncommutative field theory is presented, with emphasis on how they may be understood as states of D-branes in Type II superstring theory. Both scalar field theory and gauge theory on Moyal spaces are extensively studied. Instantons in Yang-Mills theory on the two-dimensional noncommutative torus and the fuzzy sphere are also constructed. In some instances the connection to D-brane physics is provided by a mapping of noncommutative solitons into K-homology.

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1 Introduction and background from string theory

These lecture notes provide an introduction to some selected topics in noncommutative field theory that are motivated by the interaction between string theory and noncommutative geometry. The material is intended to be geared at an audience consisting of graduate students and beginning postdoctoral researchers in mathematics, and in general any mathematician interested in how string theory crops up in certain mathematical settings. The material also serves as an introduction to physicists into some of the more formal aspects of noncommutative field theory, revolving primarily around the geometric structure of their classical solutions and the mathematical interpretation of noncommutative solitons as D-branes.

The main theme that we will focus on revolves around the scenario that string theory with D-branes in the presence of "background fields" leads to noncommutative geometries on the worldvolumes of the branes. Some familiarity with this notion along with the basics of noncommutative geometry, the standard examples of noncommutative spaces, and the rudiments of K-theory are assumed, as they were discussed in other minicourses at the workshop. Nevertheless, to set the stage and terminology, we will take a short mathematical tour in this section through this story and define the relevant concepts in the correspondence to make our presentation as self-contained as possible. Throughout we will deal only with *classical* aspects of string theory and noncommutative field theory, and all of our definitions and explanations are to be understood in this context alone. Quantization introduces many complicated technical aspects that are out of the scope of these introductory notes.

We begin by explaining some basic concepts from classical string theory. Let X be an oriented Euclidean spin manifold with Riemannian metric G. We call X the *spacetime* or the *target space*. Let Σ be a Riemann surface. A *string* is a harmonic map $x : \Sigma \to X$, i.e. an immersion of the surface Σ in X of minimal area in the induced metric $x^*(G)$. The surface Σ is called the *worldsheet* of the string. The harmonic property can be described in the usual way by a variational principle based on a σ-model on Σ with target space X. The string is said to be of

Type II if Σ is oriented, and of *Type I* if Σ is non-orientable. In the following we will only deal with Type II strings. A string is *closed* if its worldsheet is closed, $\partial\Sigma = \emptyset$, while it is *open* if its worldsheet has boundary, $\partial\Sigma \neq \emptyset$. The spin condition on X is assumed so that we can define spinors and ultimately *superstrings*, but we shall not go into any aspects of supersymmetry here.

In the simplest instance, a *D-brane* may be defined as a closed oriented submanifold $W \subset X$ which can be used as a boundary condition for open strings. This means that in the presence of D-branes the admissible open strings are the *relative* harmonic maps $x : (\Sigma, \partial\Sigma) \to (X, W)$. The submanifold W is called the *worldvolume* of the D-brane. Not all submanifolds are allowed as viable D-brane worldvolumes. For instance, a consistent choice of boundary conditions must preserve the fundamental conformal invariance of the string theory. Determining the allowed D-branes in a given spacetime X is an extremely difficult problem which requires having the quantum field theory of the worldsheet σ-model under control. For example, the cancellation of worldsheet anomalies for Type II strings requires W to be a spin^c manifold [29].

Let us next introduce the concept of a *supergravity background field*. In addition to the metric G on X, we assume the presence of an additional geometrical entity called the *Neveu-Schwarz B-field*. It is a two-form $B \in \bigwedge^2(T^*X)$, which we will locally regard as a skew-symmetric linear map $B_p : T_pX \to T_p^*X$ for $p \in X$. The B-field has curvature $H = \mathrm{d}B \in \bigwedge^3(T^*X)$ and characteristic class $[H] \in \mathrm{H}^3(X, \mathbb{Z})$. Of course as $H = \mathrm{d}B$ is an exact three-form it defines a trivial class in de Rham cohomology, but there can be torsion and other effects which yield a non-trivial characteristic class. The H-field is constrained to obey the *supergravity equation*

$$R(G) = \tfrac{1}{4}\, H \circ G^{-1} \neg H \ , \tag{1.1}$$

where $R(G)$ is the Ricci curvature two-form of the metric G and $\neg$ denotes contraction. Similarly to the B-field, the metric here and throughout is regarded locally as a symmetric non-degenerate homomorphism $G_p : T_pX \to T_p^*X$ and likewise $H_p : T_pX \to \bigwedge^2(T_p^*X)$ for $p \in X$. This equation ties the characteristic class of the B-field to the curvature of the spacetime X. The *semi-classical limit* is the one in which X "approaches" flat space, i.e. $R(G) \to 0$, and B becomes topologically trivial.

The important point is that both open and closed strings feel the presence of H, but in very different ways. Closed strings only "see" the cohomology class $[H]$. According to the supergravity equation (1.1), $[H] \to 0$ in the semi-classical limit. Thus a consistent semi-classical treatment of closed strings will be insensitive to the presence of a B-field. Aspects of the noncommutative geometry of closed strings can be found in [19, 30, 31, 59, 60, 54]. They will not be covered in these notes. In contrast, open strings are sensitive to a concrete choice of B with $H = \mathrm{d}B$. The induced two-form $x^*(B) \in \bigwedge^2(T^*\Sigma)$ does not vanish in the limit $R(G) \to 0$ and we can now explore the possibility of analysing the string geometry semi-classically in the background B-field, whereby one should be able to say some concrete things.

Consider a D-brane with embedding $\zeta : W \to X$. By a slight abuse of notation, we will denote the pullback $\zeta^*(B)$ of the B-field also by the symbol B, as this shouldn't cause any confusion in the following. If this pullback is non-degenerate, then we can define the *Seiberg-Witten bivector* [81]

$$\theta = \left(B + G\, B^{-1}\, G\right)^{-1} \ , \tag{1.2}$$

which is regarded locally as a non-degenerate skew-symmetric linear map $\theta_p : T_p^*X \to T_pX$ for $p \in X$. In the formal "limit" $B \to \infty$, the Seiberg-Witten bivector is $\theta = B^{-1}$. Thus when in

addition $H = \mathrm{d}B = 0$, as happens in the semi-classical flat space limit, the B-field defines a symplectic structure on W and θ is the Poisson bivector corresponding to the symplectic two-form B. If B is degenerate (and hence *not* symplectic), then under favourable circumstances θ will still be a Poisson bivector. One only requires that the Jacobi identity for the corresponding Poisson brackets be fulfilled. This is equivalent to the closure condition $\mathrm{d}B = 0$ only when B is non-degenerate. The quantum theory of the open strings attached to the D-brane tells us that we should *quantize* this Poisson geometry [7, 20, 78, 81]. This leads to a string theoretic picture of the Kontsevich deformation quantization of Poisson manifolds [52]. An explicit realization of this picture is provided by the Cattaneo-Felder topological σ-model [18].

When $\mathrm{d}B \neq 0$ new phenomena occur. One encounters generalizations of ordinary Poisson structures, variations of quantum group algebras, and the like [3]. The generic situation leads to non-associative deformations, which in some instances can still be handled by the realization that they define A_∞-homotopy associative structures [23]. But there is no general notion of quantization for such geometries. We will therefore continue to work with the limits described above wherein one obtains true symplectic geometries on the worldvolumes of D-branes. This sequence of limits is often referred to as the *Seiberg-Witten limit* [81].

A D-brane, and more generally collections of several D-branes, also has much more structure to it than what we have described thus far. The most primitive definition we can take of a D-brane is as a *Baum-Douglas K-cycle* (W, E, ζ) in X [11], where W is a spin^c manifold, $\zeta : W \to X$ is a continuous map and $E \to W$ is a complex vector bundle called the *Chan-Paton bundle*. The description of D-branes in terms of K-cycles and K-homology [8, 46, 74, 87] will be central to our analysis later on. If we equip E with a smooth connection, then we can define a Yang-Mills gauge theory on W. We may also define more general field theories on W by considering smooth sections of this bundle (and other canonically defined bundles over W). The connections and sections in this context are referred to as *worldvolume fields*. The semi-classical motion of the D-brane is thereby described dynamically by a *worldvolume field theory* which is induced by the quantum theory of open strings on the brane. If the geometry of the D-brane is quantized in the manner explained above, then one finds a *noncommutative field theory* on the brane worldvolume W in the Seiberg-Witten limit [81]. In the particular case of Yang-Mills theory, the deformation gives rise to a *noncommutative gauge theory*.

In what follows, for us the interesting aspects of these noncommutative worldvolume field theories will lie in the property that they possess novel classical solitonic solutions which have no counterparts in ordinary field theory [35]. In many instances these solutions can themselves be interpreted as D-branes [1, 24, 47, 48, 90, 39, 58, 72, 82], quite unlike the usual worldvolume field theories for $B = 0$. Field theoretical constructions of BPS soliton solutions in noncommutative supersymmetric Yang-Mills theory, and their applications to D-brane dynamics, can be found in [39, 32, 44, 88] (see [33, 34] for BPS soliton solutions in other noncommutative field theories). The crucial point is that the Seiberg-Witten limit still retains a lot of stringy information, in contrast to the usual field theoretic or point particle limits of string theory. We can thus use the solitons of noncommutative field theory to teach us about aspects of D-branes. For example, they can provide insights into what sort of worldvolume geometries live in a given spacetime X. Moreover, their eventual quantization (which will not be covered here) could teach us a lot about the nonperturbative structure of quantum string theory. The purpose of these lecture notes is to explain this correspondence in some specific spacetimes X, and to illustrate how the techniques

of noncommutative geometry can be used to construct the appropriate noncommutative field configurations. The classical noncommutative solitons will then admit an interpretation in terms of "branes within branes" [25] and are built solely from the properties of noncommutative field theory. A key point in our analysis will be the unveiling of the connections with the K-theory classification of D-brane charges [64, 91, 49, 67].

The outline of the remainder of these notes is as follows. In Section 2 we describe D-branes in flat Euclidean space and the corresponding noncommutative field theories. This section contains most of the elementary definitions used throughout these notes. In Section 3 we construct scalar field solitons on Moyal spaces in these settings, and give their interpretations as D-branes through an intimate connection to K-homology. In Section 4 we proceed to noncommutative gauge theory and explicitly construct instanton solutions in the two-dimensional case. In Section 5 we look at D-branes whose worldvolume is a noncommutative torus and compare the instantons in this case with those of the Moyal space. Finally, in Section 6 we consider D-branes in curved spaces described by group manifolds and the ensuing fuzzy spaces which describe the quantized worldvolume geometries, looking in particular at the classic example of fuzzy spheres in $X = \mathrm{SU}(2) \cong \mathrm{S}^3$.

2 Euclidean D-branes

We will spend most of our initial investigation working in the simplest cases of flat target spaces with $R(G) = H = 0$. Then we are automatically in the semi-classical regime of the string theory and we can proceed straightforwardly with the quantization of the brane worldvolume geometries as explained in Section 1. In this section we begin with some elementary definitions and then proceed to describe the construction of noncommutative field theories on the quantized worldvolumes. Extensive reviews of these sorts of noncommutative field theories with exhaustive lists of references can be found in [26, 51, 86].

2.1 Moyal spaces

We consider spacetimes which are either a d-dimensional Euclidean space $X = \mathbb{R}^d$ or a d-dimensional torus $X = \mathrm{T}^d$ for some integer $d \geq 2$. Let $2n \leq d$, and consider a D-brane localized along a $2n$-dimensional hyperplane $\mathbb{V}_{2n} \subset X$ with tangent space $T\mathbb{V}_{2n}$. Let $\theta : T^*\mathbb{V}_{2n} \to T\mathbb{V}_{2n}$ be a skew-symmetric non-degenerate linear form. It may be represented by a skew-symmetric $2n \times 2n$ constant real matrix $\theta = (\theta^{ij})_{1 \leq i,j \leq 2n}$ of maximal rank $2n$. There exists a linear transformation on $\mathbb{V}_{2n} \to \mathbb{V}_{2n}$ which brings θ into its canonical Jordan normal form

$$\theta = \begin{pmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \theta_n \\ -\theta_n & 0 \end{pmatrix} . \tag{2.1}$$

For definiteness we will take $\theta_k > 0$ for all $k = 1, \ldots, n$. It is straightforward to extend all of our considerations below to matrices θ of rank $< 2n$ (and in particular to odd-dimensional hyperplanes), but for simplicity we will work only with non-degenerate θ.

We now define the appropriate quantization of the D-brane worldvolume $\mathbb{V}_{2n}$ in these instances. Throughout we will denote $\mathrm{i} := \sqrt{-1}$.

Definition 2.1. Let $F(\mathbb{V}_{2n}) = \mathbb{C}\langle \mathbb{1}, x^1, \ldots, x^{2n} \rangle$ be the free unital algebra on $2n$ generators $x^1, \ldots, x^{2n}$. Let $I_\theta(\mathbb{V}_{2n})$ be the two-sided ideal of $F(\mathbb{V}_{2n})$ generated by the $n\,(2n-1)$ elements

$x^i\,x^j - x^j\,x^i - \mathrm{i}\,\theta^{ij}\,\mathbb{1}$, $1 \le i < j \le 2n$. The *Moyal $2n$-space* $\mathbb{V}^\theta_{2n}$ is the polynomial algebra $\mathbb{V}^\theta_{2n} = F(\mathbb{V}_{2n}) \,/\, I_\theta(\mathbb{V}_{2n})$.

The Moyal $2n$-space will be loosely regarded as the algebra of "functions generated by the coordinates" $x^1, \dots, x^{2n}$ satisfying the commutation relations of a degree n Heisenberg algebra

$$\big[x^i\,,\,x^j\big] := x^i\,x^j - x^j\,x^i = \mathrm{i}\,\theta^{ij}\,\mathbb{1}\ . \tag{2.2}$$

In the normal form (2.1), the only non-vanishing commutation relations are

$$\big[x^{2k-1}\,,\,x^{2k}\big] = \mathrm{i}\,\theta_k\,\mathbb{1}\ , \qquad k = 1, \dots, n\ . \tag{2.3}$$

We will also formally regard $\mathbb{V}^\theta_{2n}$ as a completion of the polynomial algebra of Definition 2.1. There are various technical complications with this since the polynomial algebra is really an algebra of differential operators and so it has no completion. We will not concern ourselves with this issue. Our definition is such that the "commutative limit" $\mathbb{V}^{\theta=0}_{2n} = \mathcal{S}(\mathbb{V}_{2n})$ is the algebra of Schwartz functions on $\mathbb{V}_{2n} \to \mathbb{C}$, i.e. the algebra of infinitely differentiable functions whose derivatives all vanish at infinity faster than any Laurent polynomial. This definition can be made more precise through the formalism of deformation quantization which will be considered in Section 2.3.

2.2 Fock modules

To do concrete calculations later on, and in particular to define gauge theories, we need to look at representations of the algebra $\mathbb{V}^\theta_{2n}$. For the moment, we set $n = 1$ and $\theta := \theta_1 > 0$. In higher dimensions we can then "glue" the n independent skew-blocks in (2.1) together, as we explain later on. Starting from the generators x^1, x^2 we introduce the formal complex linear combinations

$$a = \tfrac{1}{\sqrt{2\,\theta}}\,\big(x^1 + \mathrm{i}\,x^2\big)\ , \qquad a^\dagger = \tfrac{1}{\sqrt{2\,\theta}}\,\big(x^1 - \mathrm{i}\,x^2\big)\ . \tag{2.4}$$

Then the Heisenberg commutation relation (2.3) is equivalent to

$$\big[a\,,\,a^\dagger\big] = \mathbb{1}\ . \tag{2.5}$$

Consider the separable Hilbert space $\mathcal{F} := \ell^2(\mathbb{N}_0)$ with orthonormal basis e_n, $n \in \mathbb{N}_0$. The dual space $\mathcal{F}^*$ has corresponding basis e^*_n with the canonical dual pairing

$$\big\langle e^*_n\,,\,e_m\big\rangle = \delta_{nm}\ . \tag{2.6}$$

Operators from $\mathcal{F} \to \mathcal{F}$ are elements of the endomorphism algebra $\mathrm{End}(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{F}^*$.

Definition 2.2. The Hilbert space $\mathcal{F}$ is a finitely generated left $\mathbb{V}^\theta_2$-module, called the *Fock module*, with left action $\mathbb{V}^\theta_2 \times \mathcal{F} \to \mathcal{F}$ given by

$$a \cdot e_n = \sqrt{n}\ e_{n-1}\ , \qquad a^\dagger \cdot e_n = \sqrt{n+1}\ e_{n+1}\ .$$

To ease notation, we will not distinguish between the algebra $\mathbb{V}^\theta_2$ and its representation as operators in $\mathrm{End}(\mathcal{F})$ acting on $\mathcal{F}$.

Remark 2.1. By the Stone-von Neumann theorem, $\mathcal{F}$ is the unique irreducible representation of the Heisenberg commutation relations. There is a natural isomorphism $\mathcal{F} \cong \mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$ obtained by representing $x^2 =: M_x$ as multiplication of a function by $x \in \mathbb{R}$ and x^1 as the differential operator $\mathrm{i}\,\theta\,\frac{\mathrm{d}}{\mathrm{d}x}$. This is known as the *Schrödinger representation*.

With this action one has

$$
\begin{aligned}
a \cdot e_0 &= 0 \,, \\
\left(a^\dagger\, a\right) \cdot e_n &= n\ e_n \,, \\
e_n &= \tfrac{1}{\sqrt{n!}} \left(a^\dagger\right)^n \cdot e_0 \,.
\end{aligned}
\tag{2.7}
$$

Thus the orthonormal basis e_n forms the complete set of eigenvectors for the action of the element $a^\dagger\, a \in \mathbb{V}_2^\theta$ and is called the *number basis*. The last identity in (2.7) implies that this is the natural basis for a given representation of $a^\dagger$ on some fixed vector $e_0 \in \mathcal{F}$. For definiteness we will mostly present our computations in the number basis, but it is possible to reformulate everything in a basis independent way.

Proposition 2.1. *The Fock module $\mathcal{F}$ is projective.*

Proof. The endomorphism

$$\Pi_n := e_n \otimes e_n^*$$

is the orthogonal projection of $\mathcal{F}$ onto the one-dimensional subspace spanned by the vector $e_n \in \mathcal{F}$. The operators Π_n, $n \in \mathbb{N}_0$ generate a complete system of mutually orthogonal projectors with

$$\Pi_n\, \Pi_m = \delta_{nm}\ \Pi_n \ , \qquad \sum_{n\in\mathbb{N}_0} \Pi_n = \mathbb{1} \ .$$

For each $n \in \mathbb{N}_0$, this determines an orthogonal decomposition for the left action of the algebra $\mathbb{V}_2^\theta$ on $\mathcal{F}$ given by

$$\mathbb{V}_2^\theta = \Pi_n \cdot \mathbb{V}_2^\theta \ \oplus\ (\mathbb{1} - \Pi_n) \cdot \mathbb{V}_2^\theta \ .$$

There is a natural isomorphism $\mathcal{F} \cong \Pi_n \cdot \mathbb{V}_2^\theta$ given by $\Pi_n \cdot f \mapsto f \cdot e_n \in \mathcal{F}$ for $f \in \mathbb{V}_2^\theta$. Thus $\mathcal{F}$ is projective. □

Remark 2.2. By formally iterating the orthogonal decomposition above we have

$$\mathbb{V}_2^\theta = \bigoplus_{n\in\mathbb{N}_0} \Pi_n \cdot \mathbb{V}_2^\theta$$

with $\Pi_n \cdot \mathbb{V}_2^\theta \cong \mathcal{F}$ for each $n \in \mathbb{N}_0$. Heuristically, this means that the Fock module is the analog of a single "point" on the noncommutative space $\mathbb{V}_2^\theta$. This is analogous to what occurs in the commutative situation, whereby any function f can be decomposed formally as $f(x) = \int \mathrm{d}y\ \delta(x-y)\, f(y)$ with the evaluation maps $\delta_x(f) = f(x)$ being the analogs of the projectors Π_n above.

Finally, the generalization to higher dimensions $n > 1$ is obtained by defining

$$a_k = \tfrac{1}{\sqrt{2\,\theta_k}} \left(x^{2k-1} + \mathrm{i}\, x^{2k}\right) \ , \quad a_k^\dagger = \tfrac{1}{\sqrt{2\,\theta_k}} \left(x^{2k-1} - \mathrm{i}\, x^{2k}\right) \tag{2.8}$$

for each $k = 1, \ldots, n$. Then the non-vanishing commutation relations are given by

$$\left[a_k \, , \, a_l^\dagger\right] = \delta_{kl} \, \mathbb{1} \ . \tag{2.9}$$

In this case a right module over the algebra $\mathbb{V}^\theta_{2n}$ is obtained by taking n independent copies of the basic Fock module above with $\mathcal{F}^{\oplus n} \cong \ell^2(\mathbb{N}_0^n) \cong \mathrm{L}^2(\mathbb{R}^n, \mathrm{d}x)$. By the Hilbert hotel argument there is a natural isomorphism $\mathcal{F}^{\oplus n} \cong \mathcal{F}$.

Remark 2.3. The commutation relations (2.9) show that all algebras $\mathbb{V}^\theta_{2n}$ for $\theta_k \neq 0$ are isomorphic and one can simply scale away the noncommutativity parameters θ_k to 1, as the redefinition (2.8) essentially does. At times it will be convenient to keep θ in as an explicit parameter for comparison of the new phenomena in noncommutative field theories to ordinary field theories.

2.3 Deformation quantization

The algebra $\mathbb{V}^\theta_{2n}$ can be regarded as a deformation quantization of the algebra of Schwartz functions $\mathcal{S}(\mathbb{V}_{2n})$ on the *ordinary* hyperplane $\mathbb{V}_{2n}$ in the standard way [12] with respect to the constant symplectic two-form $\omega = \theta^{-1}$. This is common practise in the string theory literature. Although it will not be used extensively in what follows, we will briefly describe here the basic features of the approach. For most of these notes we will stick to the more abstract setting with $\mathbb{V}^\theta_{2n}$ realized as operators on the Fock module $\mathcal{F}$ (or any other module), as everything can be straightforwardly constructed in this setting. The deformation quantization approach will only be used occasionally when it can provide a useful way of envisaging the "profiles" of noncommutative fields.

Consider a polynomial function $f : \mathbb{V}_{2n} \to \mathbb{C}$ with Fourier transform $\tilde{f} : T\mathbb{V}_{2n} \to \mathbb{C}$ defined through

$$f(x) = (2\pi)^{-2n} \int_{T\mathbb{V}_{2n}} \mathrm{d}k \ \tilde{f}(k) \ \exp\big(\mathrm{i}\,\langle k, x\rangle\big) \ , \tag{2.10}$$

where $\langle k, x\rangle := \sum_{i=1}^{2n} k_i \, x^i$ with $k = (k_1, \ldots, k_{2n}) \in T\mathbb{V}_{2n}$ and $x = (x^1, \ldots, x^{2n}) \in \mathbb{V}_{2n}$. Here we will distinguish between local coordinates x^i on $\mathbb{V}_{2n}$ and the generators $\hat{x}^i$ of the noncommutative algebra $\mathbb{V}^\theta_{2n}$ by drawing a hat over the latter symbols. Thus $[\hat{x}^i, \hat{x}^j] = \mathrm{i}\,\theta^{ij}\,\mathbb{1}$. We will usually consider $\mathbb{V}^\theta_{2n}$ in its concrete realization as linear operators in $\mathrm{End}(\mathcal{F})$ acting on the Fock module, but this is not necessary and the following construction also works at a more abstract level [86].

The *Weyl symbol* $\Omega(f) \in \mathbb{V}^\theta_{2n}$ is defined by

$$\Omega(f) = (2\pi)^{-2n} \int_{T\mathbb{V}_{2n}} \mathrm{d}k \ \tilde{f}(k) \ \exp\big(\mathrm{i}\,\langle k, \hat{x}\rangle\big) \ . \tag{2.11}$$

Heuristically, the Weyl symbol of $f(x)$ is obtained by substituting in the generators of $\mathbb{V}^\theta_{2n}$ to associate to the function a *noncommutative field* $\Omega(f) = f(\hat{x})$. This definition extends to Schwartz functions $f \in \mathcal{S}(\mathbb{V}_{2n})$, for which $\Omega(f)$ is a compact operator when acting on $\mathcal{F}$. The compact operators form a dense domain $\mathcal{K} = \mathcal{K}(\mathcal{F})$ in the endomorphism algebra $\mathrm{End}(\mathcal{F})$. The exponential function of algebra elements appearing in eq. (2.11) is defined formally through its power series expansion. It implies a particular ordering of the noncommutative variables called *Weyl ordering*.

The map $f \mapsto \Omega(f)$ is invertible. Its inverse allows one to go the other way and associate functions to noncommutative fields. The *Wigner function* $\Omega^{-1}(O) \in \mathcal{S}(\mathbb{V}_{2n})$ of an element $O \in \mathcal{K}$ is given by

$$\Omega^{-1}(O)(x) = \big|\mathrm{Pf}(2\pi\,\theta)\big|^{-1} \int_{T\mathbb{V}_{2n}} \mathrm{d}k \ \exp\big(-\,\mathrm{i}\,\langle k,x\rangle\big) \ \mathrm{Tr}\ \big[O\ \exp\big(\mathrm{i}\,\langle k,\hat{x}\rangle\big)\big]\ , \tag{2.12}$$

where the symbol Pf denotes the Pfaffian of an antisymmetric matrix and Tr is the trace defined on $\mathcal{K}$. It follows that the Weyl symbol determines a vector space isomorphism between appropriate subspaces of functions on $\mathbb{V}_{2n}$ and dense domains of elements in $\mathbb{V}_{2n}^{\theta}$ in an appropriate Fréchet algebra topology. This point of view is also useful for adding further structure to the algebra $\mathbb{V}_{2n}^{\theta}$. For instance, the *trace* Tr on $\mathbb{V}_{2n}^{\theta}$ is defined for $O \in \mathcal{K}$ by

$$\mathrm{Tr}\,(O) = \int_{\mathbb{V}_{2n}} \mathrm{d}x\ \Omega^{-1}(O)(x)\ . \tag{2.13}$$

However, the linear mapping $f \mapsto \Omega(f)$ is *not* an algebra isomorphism. This fact can be used to *deform* the pointwise multiplication of functions on $\mathbb{V}_{2n}$ and define the *Moyal star-product* by

$$f \star g = \Omega^{-1}\big(\Omega(f)\,\Omega(g)\big) \tag{2.14}$$

for $f,g \in \mathcal{S}(\mathbb{V}_{2n})$. For the domains of functions we are interested in, a convenient explicit expression for the star-product is

$$(f\star g)(x) = (2\pi)^{-2n} \int_{T\mathbb{V}_{2n}} \mathrm{d}k \int_{\mathbb{V}_{2n}} \mathrm{d}y\ f\big(x+\tfrac{1}{2}\,\theta\,k\big)\ g\big(x+y\big)\ \exp\big(\mathrm{i}\,\langle k,y\rangle\big) \tag{2.15}$$

where $(\theta\,k)^i := \sum_{j=1}^{2n} \theta^{ij}\,k_j$. With this representation the star-product of two Schwartz functions is again a Schwartz function. There are other commonly used explicit expressions for the star-product in the string theory literature, such as a Fourier integral representation or a formal asymptotic expansion using a bidifferential operator [86], but these formulas do not necessarily return Schwartz functions. Let us conclude the present discussion with a class of examples that will be relevant to our later constructions.

Example 2.1. For $x=(x^1,x^2) \in \mathbb{V}_2$ with $|x|^2 := (x^1)^2+(x^2)^2$ one can straightforwardly compute the basic Gaussian Wigner function [35, 56, 57]

$$\Omega^{-1}(e_0\otimes e_0^*)(x) = 2\,\mathrm{e}^{\,-|x|^2/\theta}\ .$$

More generally, for any $n>m$ one finds the Wigner functions

$$\Omega^{-1}(e_n\otimes e_m^*)(r,\vartheta) = 2\,(-1)^m\,\sqrt{\tfrac{m!}{n!}}\ \Big(\tfrac{2\,r^2}{\theta}\Big)^{\frac{n-m}{2}}\ \mathrm{e}^{\,\mathrm{i}\,(n-m)\,\vartheta}\ \mathrm{e}^{\,-r^2/\theta}\ L_m^{n-m}\big(\tfrac{2\,r^2}{\theta}\big)$$

where $(r,\vartheta) \in [0,\infty)\times[0,2\pi)$ are plane polar coordinates on $\mathbb{V}_2$ and

$$L_k^j(t) = \sum_{l=0}^{k} (-1)^l\,\binom{k+j}{k-l}\ \frac{t^l}{l!}$$

for $j,k \in \mathbb{N}_0$ are the associated Laguerre polynomials. It is an instructive exercise to check, using the explicit integral representation (2.15), that the star-products of these functions obey the appropriate orthonormality relations required for both (2.6) and (2.14) to hold.

2.4 Derivations

The infinitesimal action of the translation group of $\mathbb{V}_{2n}$ induces automorphisms $\partial_i : \mathbb{V}^\theta_{2n} \to \mathbb{V}^\theta_{2n}$, $i = 1, \dots, 2n$. On generators they are given by

$$\partial_i\big(x^j\big) = \delta_i{}^j \ . \tag{2.16}$$

With this definition one can verify the Leibniz rule, so that the automorphisms ∂_i, $i = 1, \dots, 2n$ define a set of *derivations* of the algebra $\mathbb{V}^\theta_{2n}$. One also finds the expected commutation relations of the translation group

$$\big[\partial_i \, , \, \partial_j\big] = 0 \ . \tag{2.17}$$

It is possible to modify these relations to give a representation $[\partial_i, \partial_j] = -\,\mathrm{i}\,\Phi_{ij}\,\mathbb{1}$ of the centrally extended translation group of $\mathbb{V}_{2n}$ without affecting any of our later considerations [6], but we will stick to the setting of eq. (2.17) for simplicity.

By using the Heisenberg commutation relations (2.2), one finds for the representation of $\mathbb{V}^\theta_{2n}$ on the Fock module $\mathcal{F}$ that the derivations can be represented as *inner* automorphisms

$$\partial_i(f) = \mathrm{i}\,\sum_{j=1}^{2n}\big(\theta^{-1}\big)_{ij}\,\big[x^j \, , \, f\big] \tag{2.18}$$

for $f \in \mathbb{V}^\theta_{2n}$. Moreover, they induce the Weyl symbols of ordinary coordinate derivatives $\partial F/\partial x^i$ of Schwartz functions F through

$$\Omega\big(\tfrac{\partial F}{\partial x^i}\big) = \partial_i\big(\Omega(F)\big) \ . \tag{2.19}$$

Finally, one can show that eq. (2.13) defines an *invariant* trace for the action of the translation group since

$$\mathrm{Tr}\,\big(\partial_i(O)\big) = 0 \tag{2.20}$$

for any $O \in \mathcal{K}$, which is equivalent to the usual formula for integration by parts of Schwartz functions on $\mathbb{V}_{2n}$.

We now have all the necessary ingredients to study a broad class of field theories on the noncommutative space $\mathbb{V}^\theta_{2n}$. Elements of the noncommutative algebra provide noncommutative fields, the invariant trace gives us an integral, and the derivations introduced above yield derivatives. We begin this investigation of noncommutative field theories on the Euclidean D-brane worldvolume $\mathbb{V}_{2n}$ in the next section.

3 Solitons on $\mathbb{V}^\theta_{2n}$

In this section we will study some elementary noncommutative *scalar* field theories on the worldvolume $\mathbb{V}_{2n}$. We will construct two broad classes of noncommutative solitons and describe the rich geometric structure of the corresponding moduli spaces. We shall then demonstrate that these solutions naturally define elements of analytic K-homology [92, 63, 46], which leads into their worldvolume interpretation as D-branes in Type II string theory. Some reviews of noncommutative solitons in the contexts described here can be found in [43, 45, 77].

3.1 Projector solitons

Let $\mathrm{u}(\mathbb{V}^\theta_{2n})$ be the Lie algebra of *Hermitean* elements in the noncommutative space, i.e. the set of $\phi \in \mathbb{V}^\theta_{2n}$ for which the corresponding endomorphisms of the Fock module are Hermitean operators with respect to the underlying Hilbert space structure of $\mathcal{F}$, or equivalently the corresponding Wigner functions $\Omega^{-1}(\phi) : \mathbb{V}_{2n} \to \mathbb{R}$ are real-valued. Let $V : \mathbb{R} \to \mathbb{R}$ be a polynomial function. We may then define an *action functional* $S : \mathrm{u}(\mathbb{V}^\theta_{2n}) \to \mathbb{R}$ by

$$
\begin{aligned}
S(\phi) &:= \mathrm{Tr}\Big(\frac{1}{2}\sum_{i=1}^{2n} \partial_i(\phi)\,\partial_i(\phi) + V(\phi)\Big) \\
&= \mathrm{Tr}\Big(-\frac{1}{2}\sum_{i,j,k=1}^{2n} \left(\theta^{-1}\right)_{ij}\left(\theta^{-1}\right)_{ik}\left[x^j\,,\,\phi\right]\left[x^k\,,\,\phi\right] + V(\phi)\Big)\ ,
\end{aligned} \tag{3.1}
$$

when this expression makes sense. An action of this form describes the dynamics of D-branes in *Type IIA* string theory.

We now apply the variational principle. Identify the tangent space $T\mathbb{V}_{2n}$ with the hyperplane $\mathbb{V}_{2n}$ itself. For any $\alpha \in \mathrm{u}(\mathbb{V}^\theta_{2n})$ and $t \in \mathbb{R}$, we may then compute the variation of the action functional (3.1) by using invariance of the trace to get

$$
\frac{\delta}{\delta\phi}S(\phi) := \frac{\mathrm{d}}{\mathrm{d}t}S(\phi + t\,\alpha)\Big|_{t=0} = -\sum_{i=1}^{2n}(\partial_i \circ \partial_i)(\phi) + V'(\phi)\ . \tag{3.2}
$$

Setting this equal to 0 thereby gives the *equation of motion*

$$
V'(\phi) = \sum_{i=1}^{2n}(\partial_i \circ \partial_i)(\phi)\ . \tag{3.3}
$$

We are interested in special classes of solutions to these equations.

Definition 3.1. A *soliton* on $\mathbb{V}^\theta_{2n}$ is a solution $\phi \in \mathrm{u}(\mathbb{V}^\theta_{2n})$ of the equation of motion (3.3) for which the action functional $S(\phi)$ is well-defined and finite.

Naively, it seems to be very simple to construct soliton solutions to the equations (3.3). If λ is a critical point of the polynomial V, i.e. $V'(\lambda) = 0$, then an obvious solution is the *constant* solution $\phi_0 = \lambda\,\mathbb{1}$. However, this solution is not trace-class and so the action $S(\phi_0)$ is is not well-defined. To obtain finite action solutions, we will first look for *static solitons* having $\partial_i(\phi) = 0$. From the inner derivation property (2.18) we see that such fields live in the center of the algebra $\mathbb{V}^\theta_{2n}$. We will soon lift this requirement and show how to extend the construction to the general case. The advantage of this restriction is that the equation of motion (3.3) for static fields takes the very simple form

$$
V'(\phi) = 0\ . \tag{3.4}
$$

Since V is a polynomial, it is easy to find finite action solutions of this equation [35].

Theorem 3.1. *Let $\lambda_1, \dots, \lambda_p$ be the critical points of the polynomial function $V(\lambda)$. Then to each collection $\mathsf{P}_1, \dots, \mathsf{P}_p$ of mutually orthogonal finite rank projectors on the Fock module $\mathcal{F}$ there bijectively corresponds a static soliton*

$$
\phi_{\{\mathsf{P}_i\}} = \sum_{i=1}^{p} \lambda_i\ \mathsf{P}_i
$$

in $\mathrm{u}(\mathbb{V}^\theta_{2n})$ *of action*

$$S\big(\phi_{\{\mathsf{P}_i\}}\big) = \big|\mathrm{Pf}(2\pi\,\theta)\big|^{-1}\,\sum_{i=1}^p V(\lambda_i)~~\mathrm{Tr}\,(\mathsf{P}_i)~.$$

Remark 3.1. In these notes all projectors are assumed to be Hermitean. The soliton solution corresponding to the collection of projectors with $k :=~\mathrm{Tr}\,(\mathsf{P}_i) > 0$ for some i and $\mathrm{Tr}\,(\mathsf{P}_j) = 0$ for all $j \neq i$ is interpreted as k non-interacting solitons sitting on top of each other at the origin of $\mathbb{V}_{2n}$.

Example 3.1. The simplest projector on $\mathcal{F}$ is $\mathsf{P}_{(1)} = e_0 \otimes e_0^*$. The corresponding Wigner function is the Gaussian field computed in Example 2.1. The soliton is localized within a width $\theta^{-1/2}$ around the origin of the hyperplane. Note that this width formally goes to infinity in the commutative limit $\theta \to 0$ and the action becomes infinite. The field "delocalizes" and spreads out to the constant solution of infinite action. Since $\mathsf{P}_{(1)}$ has rank 1, it describes a single soliton at the origin. More generally, the projector

$$\mathsf{P}_{(k)} = \sum_{n=0}^{k-1} e_n \otimes e_n^*$$

has rank k and describes k solitons at the origin $x = 0$. The corresponding Wigner functions are given by combinations of Gaussian fields and Laguerre polynomials as described in Example 2.1.

3.2 Soliton moduli spaces

In D-brane physics one would like to understand what are the parameters that label inequivalent configurations modulo symmetries. These configurations live in a moduli space which determines the effective worldvolume geometries and on which we can study the effective dynamics of the branes. This is also a crucial ingredient for the eventual quantization of the systems, which would require an integration over the moduli space. Let us introduce for each $k \in \mathbb{N}$ the complex *Grassmannian*

$$\mathrm{Gr}(k, \mathcal{F}) = \mathrm{U}(\mathcal{F})\,/\,\mathrm{U}(k) \times \mathrm{U}(\mathcal{F})~, \tag{3.5}$$

where $\mathrm{U}(\mathcal{F})$ is the group of unitary endomorphisms of the Hilbert space $\mathcal{F}$ and $\mathrm{U}(k)$ is the group of $k \times k$ unitary matrices.

Proposition 3.1. *The moduli space* $\mathcal{M}^0_k(\mathbb{V}^\theta_{2n})$ *of static* k*-solitons is an infinite-dimensional Kähler manifold isomorphic to the Grassmannian*

$$\mathcal{M}^0_k\big(\mathbb{V}^\theta_{2n}\big) ~=~ \mathrm{Gr}\big(k, \mathcal{F}\big)~.$$

Proof. For static fields, the action (3.1) is invariant under the unitary transformations

$$\phi ~\longmapsto~ \mathrm{Ad}_U(\phi)$$

where $U \in \mathrm{U}(\mathcal{F})$. Any two projectors on $\mathcal{F}$ of the same rank are homotopic under this action. A projector of rank k has image which is a k-dimensional linear subspace $\mathcal{F}_k \subset \mathcal{F}$ and thus specifies

a point in the Grassmannian (3.5), where the first unitary group $\mathrm{U}(\mathcal{F})$ acts on the whole of $\mathcal{F}$, the group $\mathrm{U}(k)$ acts on the finite-dimensional subspace $\mathcal{F}_k$, and the last $\mathrm{U}(\mathcal{F})$ factor acts on the orthogonal complement $\mathcal{F}\ominus\mathcal{F}_k \cong \mathcal{F}$. Let E_k be the tautological hyperplane bundle over $\mathrm{Gr}(k,\mathcal{F})$. The inner product on E_k induces a natural metric on the determinant line bundle $\det(E_k)$. The curvature two-form of this line bundle is the natural Kähler form on the Grassmannian. □

All solitons ϕ in the infinite-dimensional moduli space $\mathcal{M}_k^0(\mathbb{V}_{2n}^\theta)$ have the same action $S(\phi) = |\mathrm{Pf}(2\pi\,\theta)|^{-1}\,k\,V(\lambda_i)$ (for some i). We can obtain a finite-dimensional soliton moduli space by "translating" the solitons obtained above away from the origin of $\mathbb{V}_{2n}$ [36]. Introduce a complex structure on $\mathbb{V}_{2n}$ with local complex coordinates $z^j = x^{2j} + \mathrm{i}\,x^{2j-1}$, $\bar z^{\bar j} = x^{2j} - \mathrm{i}\,x^{2j-1}$ for $j = 1,\dots,n$. For each $z = (z^1,\dots,z^n)$ we define the *coherent state* $\xi(z) \in \mathcal{F}$ by

$$\xi(z) = \exp\Big(\sum_{j=1}^n z^j\,a_j^\dagger\Big)\cdot e_0 \ , \tag{3.6}$$

where again the exponential operator is understood through its formal power series expansion. These vectors diagonalize the operators a_j with

$$a_j\cdot\xi(z) = z^j\ \xi(z) \ , \quad j = 1,\dots,n \ . \tag{3.7}$$

For the k-soliton solution, we place the solitons at some chosen points $z_{(0)}, z_{(1)},\dots,z_{(k-1)}$ in $\mathbb{V}_{2n}$ with $z_{(i)} = (z^1_{(i)},\dots,z^n_{(i)})$. Let $\mathsf{P}_{\{z_{(i)}\}}$ be the orthogonal projection onto the linear span of the corresponding vectors $\xi(z_{(0)}),\xi(z_{(1)}),\dots,\xi(z_{(k-1)})$. Then $\mathrm{Tr}\,(\mathsf{P}_{\{z_{(i)}\}}) = k$. The corresponding soliton solution is called a *separated soliton.*

Remark 3.2. One can compute the Wigner functions corresponding to these projectors. For $n = 1$ and $z_{(i)}$ all distinct, one finds [36]

$$\Omega^{-1}\big(\mathsf{P}_{\{z_{(i)}\}}\big)(w,\bar w) = 2\sum_{i,j=0}^{k-1}\exp\Big(-\big(\tfrac{\bar w}{\sqrt\theta} - \bar z_{(i)}\big)\big(\tfrac{w}{\sqrt\theta} - z_{(j)}\big)\Big)$$

for $(w,\bar w) \in \mathbb{V}_2$. This function has a natural interpretation in terms of separated solitons.

The operators $\mathsf{P}_{\{z_{(i)}\}}$ may be characterized as those projectors P obeying the equations

$$(\mathbb{1} - \mathsf{P})\,a_j\,\mathsf{P} = 0 \ , \quad j = 1,\dots,n \ , \tag{3.8}$$

or equivalently that the image $\mathrm{im}(\mathsf{P}) \subset \mathcal{F}$ is an invariant subspace for the collection of operators $a_1,\dots,a_n$. As we will now show, they define a finite-dimensional subspace $\mathcal{M}_k(\mathbb{V}_{2n}^\theta) \subset \mathrm{Gr}(k,\mathcal{F})$. Introduce the *Hilbert scheme* $\mathrm{Hilb}^k(\mathbb{V}_{2n})$ of k points in $\mathbb{V}_{2n} \cong \mathbb{C}^n$ as the space of ideals $\mathcal{I}$ of codimension k in the polynomial ring $\mathbb{C}[y^1,\dots,y^n]$. It is easy to see at a heuristic level how the Hilbert scheme is related to the projectors constructed above. Since $f \in \mathcal{I}$ implies that $f\,g \in \mathcal{I}$ for all polynomial functions g, the polynomials in an ideal $\mathcal{I}$ may be thought of roughly as projections on $\mathbb{C}[y^1,\dots,y^n] \to \mathcal{I}$. Conversely, if $\mathsf{P} = \mathsf{P}_{\{z_{(i)}\}}$ with all $z_{(i)}$ distinct, then the corresponding ideal $\mathcal{I}$ consists of those polynomials $f \in \mathbb{C}[y^1,\dots,y^n]$ which vanish at the loci $z_{(i)}$, i.e. $f(z^1_{(i)},\dots,z^n_{(i)}) = 0$ for each $i = 0,1,\dots,k-1$. This correspondence can be made more precise.

Theorem 3.2. *The moduli space $\mathcal{M}_k(\mathbb{V}_{2n}^\theta)$ of separated k-solitons is the Hilbert scheme*

$$\mathcal{M}_k\big(\mathbb{V}_{2n}^\theta\big) \ = \ \mathrm{Hilb}^k\big(\mathbb{V}_{2n}\big) \ .$$

Proof. We set up a one-to-one correspondence between projectors $\mathsf{P} \in \mathcal{M}_k(\mathbb{V}_{2n}^\theta)$ obeying eq. (3.8) and ideals $\mathcal{I} \in \mathrm{Hilb}^k(\mathbb{V}_{2n})$. Define for each polynomial $f \in \mathbb{C}[y^1, \dots, y^n]$ the vector

$$e_f = f\big(a_1^\dagger, \dots, a_n^\dagger\big) \cdot e_0$$

in $\mathcal{F}$. If $\mathcal{I} \subset \mathbb{C}[y^1, \dots, y^n]$ is an ideal of codimension k, then we let $\mathbb{1} - \mathsf{P}$ be the orthogonal projection of $\mathcal{F}$ onto the linear span $\bigoplus_{f \in \mathcal{I}} \mathbb{C} \cdot e_f$. Conversely, if $\mathsf{P} \in \mathcal{M}_k(\mathbb{V}_{2n}^\theta)$ we set

$$\mathcal{I} = \big\{ f \in \mathbb{C}[y^1, \dots, y^n] \bigm| \mathsf{P} \cdot e_f = 0 \big\} ,$$

which is an ideal since $\mathsf{P}\, a_j^\dagger = \mathsf{P}\, a_j^\dagger\, \mathsf{P}$ for each $j = 1, \dots, n$. □

Example 3.2. Theorem 3.2 allows us to work out some explicit soliton moduli spaces for low values of the integers k and n.

1. $\mathcal{M}_k(\mathbb{V}_2^\theta) = \mathrm{Hilb}^k(\mathbb{V}_2)$ is the k-th symmetric product orbifold $\mathrm{Sym}^k(\mathbb{V}_2) = (\mathbb{V}_2)^k / S_k$, where the symmetric group S_k acts on the soliton positions in $(\mathbb{V}_2)^k$ by permuting the entries of a k-tuple of elements of the plane $\mathbb{V}_2$. The Kähler metric inherited from the Grassmannian is smooth at the orbifold points corresponding to coinciding soliton positions in $(\mathbb{V}_2)^k$ [36, 41]. Thus noncommutativity smooths out the orbifold singularities of the symmetric product and as Kähler manifolds one has an isomorphism

$$\mathcal{M}_k\big(\mathbb{V}_2^\theta\big) = \big(\mathbb{V}_2\big)^k .$$

2. The two-soliton moduli space is [36]

$$\mathcal{M}_2\big(\mathbb{V}_{2n}^\theta\big) = \mathbb{V}_{2n} \times \mathcal{O}_{\mathbb{P}^{n-1}}\big(-1\big)$$

   where $\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to \mathbb{P}^{n-1}$ is the Hopf bundle over the complex projective space $\mathbb{P}^{n-1}$. The first factor describes the center of mass position of the soliton configuration, while the second factor is the resolution of the singularity of the moduli space for the relative position which blows up the origin into $\mathbb{P}^{n-2}$. Again the moduli space is non-singular.

3. $\mathcal{M}_k(\mathbb{V}_4^\theta) = \mathrm{Hilb}^k(\mathbb{V}_4)$ is a smooth manifold which is a crepant resolution of the singular quotient variety $\mathrm{Sym}^k(\mathbb{V}_4)$. It also arises as the moduli space of k U(1) instantons on $\mathbb{V}_4^\theta$ [16, 36]. However, while the instanton moduli space is endowed with a hyper-Kähler metric, the soliton moduli space $\mathcal{M}_k(\mathbb{V}_4^\theta)$ is only a Kähler manifold.

4. For $n > 2$ and $k > 3$ the moduli space $\mathcal{M}_k(\mathbb{V}_{2n}^\theta)$ generically contains branches of varying dimension and so is not even a manifold [66].

3.3 Partial isometry solitons

We will now construct solitons corresponding to general complex elements $\phi \in \mathbb{V}_{2n}^\theta$. We call these *complex solitons*. We use the same polynomial function V as before, but now the action functional $S : \mathbb{V}_{2n}^\theta \to \mathbb{R}$ is defined by

$$S\big(\phi\,,\,\phi^\dagger\big) = \mathrm{Tr}\,\Big(\sum_{i=1}^{2n} \partial_i\big(\phi\big)\,\partial_i\big(\phi^\dagger\big) + V\big(\phi^\dagger\phi - \mathbb{1}\big) + V\big(\phi\,\phi^\dagger - \mathbb{1}\big)\Big) . \tag{3.9}$$

Such an action describes the dynamics of D-branes in *Type IIB* string theory. As before it is straightforward to obtain soliton solutions of the equations of motion corresponding to (3.9).

Proposition 3.2. *To each partially isometric Fredholm operator on the Fock module $\mathcal{F}$ there bijectively corresponds a static complex soliton.*

Proof. Varying ϕ and $\phi^\dagger$ independently in the action functional (3.9) shows that the equations of motion for static fields are satisfied if $\phi, \phi^\dagger$ obey the defining equation

$$\phi\,\phi^\dagger\,\phi = \phi$$

for a partial isometry of $\mathcal{F}$. Equivalently, ϕ is an isometry in the orthogonal complement to a kernel and cokernel, or

$$\phi^\dagger\,\phi = \mathbb{1} - \mathsf{P}_{\ker(\phi)}\ , \quad \phi\,\phi^\dagger = \mathbb{1} - \mathsf{P}_{\mathrm{coker}(\phi)}$$

where $\mathsf{P}_{\ker(\phi)}$ and $\mathsf{P}_{\mathrm{coker}(\phi)}$ are the orthogonal projections onto the kernel and cokernel of ϕ. Substituting these expressions into the action functional (3.9) and demanding that it be finite requires that both $\ker(\phi)$ and $\mathrm{coker}(\phi)$ be finite-dimensional subspaces of $\mathcal{F}$, i.e. that ϕ be also a Fredholm operator. □

Remark 3.3. Using Remark 2.2, the finite-dimensional subspaces $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are identified with the vanishing locus of the complex soliton in the corresponding Wigner function formulation on $\mathbb{V}_{2n}$.

Definition 3.2. The *topological charge* $Q(\phi)$ of a complex soliton $\phi \in \mathbb{V}^\theta_{2n}$ is its analytic index $Q(\phi) := \mathrm{index}(\phi) = \dim\ker(\phi) - \dim\,\mathrm{coker}(\phi)$.

To explicitly construct such solitons, let $C\ell(\mathbb{V}_{2n})$ be the complex Clifford algebra of the vector space $\mathbb{V}_{2n}$ equipped with the canonical quadratic form. Let $\Delta_\pm$ be the irreducible half-spinor modules over $C\ell(\mathbb{V}_{2n})$ of ranks $r := 2^{n-1}$. The half-spinor generators are denoted $\sigma_i : \Delta_+ \to \Delta_-$, $i = 1, \dots, 2n$ and they satisfy the algebra

$$\sigma_i{}^\dagger\,\sigma_j + \sigma_j{}^\dagger\,\sigma_i \;=\; 2\,\delta_{ij}\,\mathbb{1}_r \;=\; \sigma_i\,\sigma_j{}^\dagger + \sigma_j\,\sigma_i{}^\dagger\ . \tag{3.10}$$

We introduce the operator

$$\sigma_x := \sum_{i=1}^{2n} \sigma_i \otimes x^i \tag{3.11}$$

regarded as an element $\sigma_x \in \mathrm{Hom}(\Delta_+ \otimes \mathcal{F}, \Delta_- \otimes \mathcal{F})$.

Lemma 3.1. *The operator σ_x has one-dimensional kernel and no cokernel.*

Proof. From eqs. (2.9) and (3.10) it follows that the operator (3.11) and its adjoint obey the identities

$$\begin{aligned}
\sigma_x\,\sigma_x{}^\dagger &= \sum_{i=1}^{n} \mathbb{1}_r \otimes 2\,\theta_i\,\big(a_i^\dagger\,a_i + \tfrac12\,\mathbb{1}\big) - \sum_{i,j=1}^{2n} \mathrm{i}\,\theta^{ij}\,\Sigma_{ij} \otimes \mathbb{1}\ , \\
\sigma_x{}^\dagger\,\sigma_x &= \sum_{i=1}^{n} \mathbb{1}_r \otimes 2\,\theta_i\,\big(a_i^\dagger\,a_i + \tfrac12\,\mathbb{1}\big) - \sum_{i,j=1}^{2n} \mathrm{i}\,\theta^{ij}\,\Sigma_{ij}{}^\dagger \otimes \mathbb{1}
\end{aligned}$$

where

$$\Sigma_{ij} = \tfrac{1}{4}\left(\sigma_i\,\sigma_j{}^\dagger - \sigma_j\,\sigma_i{}^\dagger\right) \ , \quad \Sigma_{ij}{}^\dagger = \tfrac{1}{4}\left(\sigma_i{}^\dagger\,\sigma_j - \sigma_j{}^\dagger\,\sigma_i\right) \ .$$

By elementary group theory, the last term in the second product is diagonalized by the lowest weight spinor ψ_0 of $\mathrm{SO}(2n)$ to $-\sum_{i=1}^n\,\theta_i\,\mathbb{1}_r\otimes\mathbb{1}$. Along with eq. (2.7), this implies that the operator σ_x has a one-dimensional kernel in $\Delta_+\otimes\mathcal{F}$ which is spanned by the vector $\psi_0\otimes e_0$. On the other hand, the right-hand side of the first product can never vanish and so the kernel of $\sigma_x{}^\dagger$ is trivial. □

Theorem 3.3. *The surjection* $\mathsf{T}\in\mathrm{Hom}(\Delta_+\otimes\mathcal{F},\Delta_-\otimes\mathcal{F})$ *defined by*

$$\mathsf{T} = \left(\sigma_x\,\sigma_x{}^\dagger\right)^{-1/2}\,\sigma_x$$

is a complex soliton of topological charge $Q(\mathsf{T})=1$*.*

Proof. By Lemma 3.1 the positive operator $\sigma_x\,\sigma_x{}^\dagger$ is invertible and so the operator T is well-defined. Furthermore, it is a partial isometry, $\mathsf{T}\,\mathsf{T}^\dagger\,\mathsf{T}=\mathsf{T}$, and has one-dimensional kernel and trivial cokernel with

$$\mathsf{T}\,\mathsf{T}^\dagger = \mathbb{1}_r\otimes\mathbb{1} \ , \quad \mathsf{T}^\dagger\,\mathsf{T} = \mathbb{1}_r\otimes\mathbb{1} - \mathsf{P}_{\ker\sigma_x}$$

implying that $\mathrm{im}(\mathsf{T})=\Delta_-\otimes\mathcal{F}$. □

Remark 3.4. The classical solution of Theorem 3.3 is interpreted as a single-soliton solution. More generally, a k-soliton solution is given by the power $(\mathsf{T})^k$ which has no cokernel and k-dimensional kernel by the Boutet de Monvel index theorem [15], i.e. $Q(\mathsf{T})^k=k$. One can also construct separated complex solitons by "translating" these operators away from the origin to points $x_{(0)},x_{(1)},\dots,x_{(k-1)}\in\mathbb{V}_{2n}$ and defining [62]

$$\mathsf{T}_{\{x_{(i)}\}} = \prod_{i=0}^{k-1}\left(\sigma_{x-x_{(i)}\,\mathbb{1}}\,\sigma_{x-x_{(i)}\,\mathbb{1}}{}^\dagger\right)^{-1/2}\sigma_{x-x_{(i)}\,\mathbb{1}} \ .$$

The analysis of the moduli space of these separated complex solitons is similar to that carried out in Section 3.2.

3.4 Topological charges

We will now derive a geometric formula for the topological charge of a complex soliton by relating the analytic index of T to a topological index [46, 63]. Let $\mathrm{S}^{2n-1}=\{x\in\mathbb{V}_{2n}\mid |x|=1\}$ be the unit sphere of odd dimension $2n-1$ in the hyperplane $\mathbb{V}_{2n}$. The restriction map $\mathbb{V}_{2n}\setminus\{0\}\to\mathrm{S}^{2n-1}$ is defined by $x\mapsto\frac{x}{|x|}$. The soliton T constructed in Theorem 3.3 can be thought of as a noncommutative version of the map $\mu:\mathrm{S}^{2n-1}\to\mathrm{GL}(r,\mathbb{C})$ defined by

$$\mu_x = \sum_{i=1}^{2n}\frac{x^i}{|x|}\,\sigma_i \ , \tag{3.12}$$

which is Clifford multiplication on the $C\ell(\mathbb{V}_{2n})$-module Δ_+ by the vector $x\in\mathbb{V}_{2n}$. To make this correspondence more precise, we have to explain what we mean by restricting an operator to a noncommutative sphere.

We begin by choosing another polarization of the Fock module $\mathcal{F}$ which can be regarded as the holomorphic version of the Schrödinger representation defined in Remark 2.1. The *Bargmann quantization* is the natural isomorphism

$$\mathcal{F} \cong \mathcal{F}_{\rm B} := {\rm Hol}\big(\mathbb{V}_{2n} \,,\, \exp(-2 \sum_{k=1}^{n} \theta_k \, |z^k|^2) \; {\rm d}z\big) \tag{3.13}$$

of $\mathbb{V}_{2n}^{\theta}$-modules obtained by representing $a_k = M_{z^k}$ as multiplication by the complex coordinate z^k and $a_k^\dagger$ as the partial differential operator $-\frac{\partial}{\partial z^k}$. In Bargmann quantization, an orthogonal basis is provided by the collection of monomials

$$z^{\vec{k}} := \prod_{i=1}^{n} \big(z^i\big)^{k_i} \ , \qquad \vec{k} = (k_1, \dots, k_n) \in \mathbb{N}_0^n \ . \tag{3.14}$$

The advantage of the diffeomorphism (3.13) is that there is a precise way to restrict vectors in $\mathcal{F}_{\rm B}$ to the sphere ${\rm S}^{2n-1}$. Using polar coordinates we decompose the canonical measure on $\mathbb{V}_{2n}$ as ${\rm d}z = r^{2n-1}\, {\rm d}r\, {\rm d}\Omega$, where $r \in \mathbb{R}_+ := [0, \infty)$ and ${\rm d}\Omega$ is the standard round measure on the unit sphere ${\rm S}^{2n-1}$. We use homogeneous complex coordinates $(z, \bar z)$ on ${\rm S}^{2n-1}$ with $\sum_{k=1}^n |z^k|^2 = 1$. The *Hardy space* ${\rm H}^2({\rm S}^{2n-1}, {\rm d}\Omega)$ is the closed Hilbert subspace of ${\rm L}^2({\rm S}^{2n-1}, {\rm d}\Omega)$ consisting of ${\rm L}^2$-functions on ${\rm S}^{2n-1}$ which admit a holomorphic extension to all of $\mathbb{V}_{2n}$. An orthogonal basis for the Hardy space ${\rm H}^2({\rm S}^{2n-1}, {\rm d}\Omega)$ is again provided by the monomials [46]

$$\varphi_{\vec{k}} := z^{\vec{k}} \ . \tag{3.15}$$

To specify the restriction of the $\mathbb{V}_{2n}^{\theta}$-module structure, we need to make sense of the action of the classical coordinates $z^k, \bar z^{\bar k}$ on ${\rm H}^2({\rm S}^{2n-1}, {\rm d}\Omega)$. Let $f : {\rm S}^{2n-1} \to \mathbb{C}$ be an ${\rm L}^2$-function. Let $P_+ : {\rm L}^2({\rm S}^{2n-1}, {\rm d}\Omega) \to {\rm H}^2({\rm S}^{2n-1}, {\rm d}\Omega)$ be the *Szegó projection* defined by

$$(P_+ f)(z) = \int_{{\rm S}^{2n-1}} {\rm d}\Omega(w, \bar w) \ \frac{f(w, \bar w)}{\Big(1 - \sum\limits_{k=1}^{n} \, z^k \, \bar w^{\bar k}\Big)^n} \ , \tag{3.16}$$

and let $M_f : {\rm H}^2({\rm S}^{2n-1}, {\rm d}\Omega) \to {\rm L}^2({\rm S}^{2n-1}, {\rm d}\Omega)$ be the operator of multiplication by f. The *Toeplitz operator* $\mathsf{T}_f : {\rm H}^2({\rm S}^{2n-1}, {\rm d}\Omega) \to {\rm H}^2({\rm S}^{2n-1}, {\rm d}\Omega)$ is then defined by

$$\mathsf{T}_f = P_+ \circ M_f \ . \tag{3.17}$$

It is the *compression* of the multiplication operator M_f to ${\rm H}^2({\rm S}^{2n-1}, {\rm d}\Omega)$. The action of the Toeplitz operators corresponding to the coordinate functions on the basis (3.15) is straightforward to work out. Let $\vec{e}_i$ denote the i-th standard unit vector in $\mathbb{R}^n$. Then for each $i = 1, \dots, n$ one has

$$\begin{aligned} \mathsf{T}_{z^i} \varphi_{\vec{k}} &= \varphi_{\vec{k} + \vec{e}_i} \ , \\ \mathsf{T}_{\bar z^{\bar\imath}} \varphi_{\vec{k}} &= \begin{cases} 0 & , \qquad k_i = 0 \\ 2\pi \, \frac{k_i}{|\vec{k}| + n - 1} \, \varphi_{\vec{k} - \vec{e}_i} & , \qquad k_i > 0 \end{cases} \end{aligned} \tag{3.18}$$

where $|\vec{k}| := \sum_{i=1}^n k_i$ for $\vec{k} \in \mathbb{N}_0^n$.

Example 3.3. It is instructive at this stage to look explicitly at the two-dimensional case $n = 1$. Then $r = 1$, $\sigma_1 = 1$ and $\sigma_2 = \mathrm{i}$ so that the standard complex soliton

$$\mathsf{T} = \big(a^\dagger\, a\big)^{-1/2}\, a =: \mathsf{S}^\dagger$$

coincides with the *shift operator* $\mathsf{S} : \mathcal{F} \to \mathcal{F}$ defined on the number basis by

$$\mathsf{S} e_m = e_{m+1} \ .$$

The shift operator is the basic partial isometry of $\mathcal{F}$ with

$$\mathsf{S}^\dagger\, \mathsf{S} = \mathbb{1} \ , \quad \mathsf{S}\, \mathsf{S}^\dagger = \mathbb{1} - e_0 \otimes e_0^* \ ,$$

and hence it has no kernel and a one-dimensional cokernel spanned by the lowest vector e_0 in the number basis of $\mathcal{F}$. The Hardy space in this instance $\mathrm{H}^2(\mathrm{S}^1, \mathrm{d}\Omega) = \mathcal{H}_+(\mathrm{S}^1)$ is the closed Hilbert subspace of $\mathrm{L}^2(\mathrm{S}^1, \mathrm{d}\Omega)$ spanned by the non-negative Fourier modes on the circle $\varphi_k = \mathrm{e}^{\,\mathrm{i}\, k\, \Omega}$, $k \geq 0$, $\Omega \in [0, 2\pi)$. This is the positive eigenspace of the Dirac operator $-\mathrm{i}\, \frac{\mathrm{d}}{\mathrm{d}\Omega}$ on S^1. By identifying the monomial φ_k with the number basis element $e_k \in \mathcal{F}$, one can identify the Toeplitz operators $\mathsf{T}_{\varphi_k} = (\mathsf{S})^k$ for $k > 0$ (which have non-trivial kernels for $k < 0$).

The correspondence between Toeplitz operators and complex solitons on $\mathbb{V}^\theta_{2n}$ given in Example 3.3 can be generalized to higher dimensions $n > 1$ by replacing the Hardy space with $\mathrm{H}^2(\mathrm{S}^{2n-1}, \mathrm{d}\Omega) \otimes \mathbb{C}^r$ (with $r := 2^{n-1}$ as before) and extending the Toeplitz operators T_f to matrix-valued L^2-functions $f : \mathrm{S}^{2n-1} \to \mathbb{M}_r(\mathbb{C})$. Then the complex soliton of Theorem 3.3 corresponds to a Toeplitz operator

$$\mathsf{T}_\mu = P_+ \circ M_\mu \tag{3.19}$$

in $\mathrm{End}(\mathrm{H}^2(\mathrm{S}^{2n-1}, \mathrm{d}\Omega) \otimes \Delta)$, where Δ is the unique irreducible spinor module of rank r over the Clifford algebra $C\ell(\mathbb{R}^{2n-1})$. This defines a bounded Fredholm operator on Hardy space and we finally arrive at our desired geometric formula for the topological charge.

Theorem 3.4. *The topological charge of the complex soliton* T *is given by the characteristic class formula*

$$Q(\mathsf{T}) = \mathrm{ch}(\mu)\big[\,\mathrm{S}^{2n-1}\,\big]$$

where

$$\mathrm{ch}(\mu) = \mu^*\Big(\sum_{j=1}^{n} \tfrac{(-1)^j}{(j-1)!}\, \omega_{2j-1}\Big)$$

and ω_i *are the standard generators of the rational cohomology* $\mathrm{H}^i(\mathrm{GL}(r, \mathbb{C}), \mathbb{Q})$.

Proof. Represent the partial isometry T in Bargmann quantization. Then the Toeplitz operator (3.19) is the image of T under the restriction map $\mathcal{F}_{\mathrm{B}} \otimes \Delta \to \mathrm{H}^2(\mathrm{S}^{2n-1}, \mathrm{d}\Omega) \otimes \Delta$. This map is not unitary, but it is a bijection and consequently $\mathrm{index}(\mathsf{T}) = \mathrm{index}(\mathsf{T}_\mu)$. The result now follows from the Boutet de Monvel index theorem [15] applied to the Toeplitz operator T_μ and the fact that the sphere S^{2n-1} has trivial Todd class. □

Remark 3.5. For $n = 1$, Theorem 3.4 reads $Q(\mathsf{T}) = \frac{1}{2\pi\,\mathrm{i}} \int_{\mathrm{S}^1} \mu^{-1}\, \mathrm{d}\mu$ and thus the topological charge of the noncommutative soliton coincides with the winding number of the function $\mu : \mathrm{S}^1 \to \mathrm{S}^1$.

3.5 Worldvolume construction

We will now demonstrate how the soliton solution constructed in Section 3.3 has a natural interpretation in terms of D-branes. The construction of these solitons is intimately related to the Atiyah-Bott-Shapiro (ABS) construction $M_\sharp\,\mathrm{Spin}(\mathbb{V}_{2n}) \to \mathrm{K}^\sharp(\mathbb{V}_{2n})$ of K-theory classes in terms of Clifford modules, whose generator is provided by Clifford multiplication (3.12). By Theorem 3.4, the topological charge of the noncommutative soliton coincides with the index of the classical ABS class of μ, whose winding number determines *D-brane charge* [91, 49, 67]. By elucidating this point we will link our solutions naturally with D-branes. Our ensuing worldvolume interpretation will thereby demonstrate the equivalence between the usual commutative and the noncommutative descriptions of D-branes, and will further provide a novel insight into the nature of the worldvolume geometries.

The basic idea behind the construction is to associate D-branes to algebras of "almost commuting" operators [46]. Let $\mathcal{B}(\mathcal{F}) \subset \mathrm{End}(\mathcal{F})$ be the C^*-algebra of bounded linear operators on the Hilbert space $\mathcal{F}$. The C^*-algebra $\mathcal{K} = \mathcal{K}(\mathcal{F})$ of compact operators on $\mathcal{F}$ is a closed ideal in $\mathcal{B}(\mathcal{F})$. The Toeplitz operators (3.17) generate a unique C^*-algebra called the *Toeplitz algebra* which we will denote by $\mathcal{A} \subset \mathcal{B}(\mathcal{F})$. To ease notation we write $X := \mathrm{S}^{2n-1}$. Let $C(X)$ be the commutative C^*-algebra of continuous complex-valued functions on X. In general, the map $C(X) \to \mathcal{A}$ defined by $f \mapsto \mathsf{T}_f$ is not an algebra homomorphism.

Proposition 3.3. *For each pair of functions $f, g \in C(X)$, the difference $\mathsf{T}_f\,\mathsf{T}_g - \mathsf{T}_{f\,g}$ in the Toeplitz algebra $\mathcal{A}$ is a compact operator on the Fock module $\mathcal{F}$.*

It follows that $[\mathsf{T}_f, \mathsf{T}_g] \in \mathcal{K}$ is compact for any $f, g \in C(X)$ [17], and compact operators are always regarded as "small" (being elements of a closed dense domain in $\mathcal{B}(\mathcal{F})$). Thus the Toeplitz algebra $\mathcal{A}$ is "almost commuting". We can identify operators which differ from one another only by a "small" perturbation by regarding them as elements of the *Calkin algebra* $\mathcal{Q}(\mathcal{F}) := \mathcal{B}(\mathcal{F}) \,/\, \mathcal{K}$ with the natural projection $\pi : \mathcal{B}(\mathcal{F}) \to \mathcal{Q}(\mathcal{F})$. The Calkin algebra is a unital C^*-algebra. In this way our explicit construction of static complex solitons in Section 3.3 leads naturally to the Brown-Douglas-Fillmore classification of essentially normal operators [17, 46].

Starting from the soliton configuration (3.19), the map $\mathsf{T}_\mu \mapsto \mu$ gives rise to a C^*-epimorphism $\beta : \mathcal{A} \to C(X) \otimes \mathbb{M}_r(\mathbb{C})$. The algebra $C(X) \otimes \mathbb{M}_r(\mathbb{C})$ is Morita equivalent to $C(X)$, and hence it will suffice to restrict our attention to the commutative algebra $C(X)$. It follows that the Toeplitz operators generate an *extension* of the commutative algebra of functions on X by compact operators, i.e. the noncommutative algebra $\mathcal{A}$ fits into a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{\beta} C(X) \longrightarrow 0 \, . \tag{3.20}$$

Exactness of the sequence follows from Proposition 3.3 and the fact that $\mathsf{T}_f\,\mathsf{T}_g - \mathsf{T}_{f\,g} \in \ker(\beta)$ for any two functions $f, g \in C(X)$.

We can introduce a set of equivalence classes of extensions (3.20) as follows. Define a map $\tau : C(X) \to \mathcal{Q}(\mathcal{F})$ called the *Busby invariant* by $\tau(f) = \pi(\mathsf{T}_f)$. By Proposition 3.3, the Busby invariant is a unital C^*-monomorphism with $\mathcal{A} = \pi^{-1}(\mathrm{im}\,\tau)$. Any extension (3.20) can be uniquely characterized by a pair $(\mathcal{H}, \tau)$, where $\mathcal{H}$ is a separable Hilbert space and $\tau : C(X) \to \mathcal{Q}(\mathcal{H})$ is a unital C^*-monomorphism [17]. On the collection of pairs $(\mathcal{H}, \tau)$, there is a natural notion of unitary (or strong) equivalence and a natural direct sum operation [17, 11, 46]. The

set $\mathrm{Ext}(C(X),\mathcal{K})$ of equivalence classes of extensions (3.20) is thus a semigroup. An extension is *trivial* if the exact sequence (3.20) splits, i.e. if the corresponding Busby invariant τ has a lift to all of $\mathcal{B}(\mathcal{H})$. The quotient of $\mathrm{Ext}(C(X),\mathcal{K})$ by trivial extensions is an abelian group which defines a dual homology theory to the K-theory of X. This is called the *analytic K-homology group* $\mathrm{K}_1^{\mathrm{a}}(X)$.

The constructions of this section bring us finally to our main result [11].

Theorem 3.5. *There is a one-to-one correspondence between static complex solitons* $\mathsf{T} \in \mathbb{V}_{2n}^{\theta}$ *with Toeplitz extension classes (3.20) in* $\mathrm{K}_1^{\mathrm{a}}(X)$ *and D-branes* (W,E,ζ) *on* X *with odd-dimensional worldvolumes* W.

Proof. Let $\Delta_W \to W$ be the spin^c bundle over the odd-dimensional spin^c manifold W. Let $\mathcal{H} = \mathrm{L}^2(W,\Delta_W \otimes E)$ be the Hilbert space of square-integrable sections of the twisted spin^c bundle over the worldvolume. Equip the Chan-Paton bundle $E \to W$ with a connection, and let W inherit the metric from X by pullback under the continuous map $\zeta : W \to X$. The corresponding twisted Dirac operator $\not{D}_E$ can be viewed as a closed unbounded operator $\not{D}_E : \mathcal{H} \to \mathcal{H}$. If the connection and metric are generic, then $\not{D}_E$ has no kernel and so the Hilbert space $\mathcal{H}$ admits an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

into the positive/negative eigenspaces $\mathcal{H}_\pm$ of the Dirac operator $\not{D}_E$.

We can represent the commutative C^*-algebra $C(W)$ of worldvolume functions on $\mathcal{H}$ by multiplication operators M_f for $f \in C(W)$. Generically the operator M_f does not preserve the subspace $\mathcal{H}_+$, but as before its compression to a Toeplitz operator $\mathsf{T}_f = P_+ \circ M_f : \mathcal{H}_+ \to \mathcal{H}_+$ does, where now P_+ is the orthogonal projection $\mathcal{H} \to \mathcal{H}_+$. As before, for $f,g \in C(W)$ the difference $\mathsf{T}_f\,\mathsf{T}_g - \mathsf{T}_{f\,g}$ is a compact operator on $\mathcal{H}_+$ and so we get a Toeplitz extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A}_W \longrightarrow C(W) \longrightarrow 0$$

of the algebra of worldvolume functions, where $\mathcal{A}_W$ is the C^*-algebra generated by T_f for $f \in C(W)$. Let $\tau_W : C(W) \to \mathcal{Q}(\mathcal{H})$ be the corresponding Busby invariant. Then by using the pullback $\zeta^* : C(X) \to C(W)$ we can define a new Busby invariant $\widetilde{\tau} := \tau_W \circ \zeta^* : C(X) \to \mathcal{Q}(\mathcal{H})$ which corresponds to an extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \widetilde{\mathcal{A}} \longrightarrow C(X) \longrightarrow 0$$

of the algebra of functions on all of X. The corresponding K-homology class in $\mathrm{K}_1^{\mathrm{a}}(X)$ is independent of the metric on X and of the choice of connection on E. □

Remark 3.6. The proof of the converse of the result proven here is beyond the scope of these notes. The one-to-one correspondence between extension classes in $\mathrm{K}_1^{\mathrm{a}}(X)$ and D-branes requires taking the quotient of the set of all Baum-Douglas K-cycles (W,E,ζ) by a collection of equivalence relations [11, 74]. These relations make good physical sense and the corresponding equivalence classes $[W,E,\zeta]$ capture the novel dynamical processes of D-brane physics [46, 8, 74, 87, 58]. Note that in this correspondence the D-brane worldvolume W need not be a submanifold of X. The problem of finding spaces X for which the generators of K-homology can be constructed from cycles of X is related to the Hodge conjecture. A projective algebraic variety that satisfies both the requirements that cycles generate K-homology [74] and the hypothesis of the Hodge

conjecture has certain restrictions on its cohomology. A Calabi-Yau threefold satisfies these conditions, and thus the K-cycles in physically viable string compactifications X correspond to D-branes whose worldvolumes are cycles of X.

Theorem 3.5 provides us with the proper perspective on the noncommutative solitons that we have constructed in this section. The solitons provide the K-homology version of the equivalence of D-brane charges in commutative and noncommutative field theories. This is asserted via the equivalence between the analytic index and the topological index via the Baum-Douglas K-cycle construction [11, 74]. The Toeplitz operators on Hardy space $\mathrm{H}^2(\mathrm{S}^{2n-1},\mathrm{d}\Omega)$ determine an algebra $\mathcal{A}$ of endomorphisms providing a non-trivial extension (3.20) by compact operators. This defines an analytic K-homology class in $\mathrm{K}_1^{\mathrm{a}}(\mathrm{S}^{2n-1})$ which is the same as the class $[\not{D}]$ determined by the Dirac operator on S^{2n-1} [11]. In particular, Theorem 3.4 now follows from the ordinary Atiyah-Singer index theorem. We may associate this class with one in degree zero K-homology which is the appropriate receptacle for the classification of D-brane charges in Type IIB string theory [91, 67]. If we work in *relative* K-homology [74], then the connecting homomorphism in the six-term exact sequence for the pair $(\mathrm{B}^{2n},\mathrm{S}^{2n-1}=\partial\mathrm{B}^{2n})$ yields an isomorphism

$$\partial\,:\,\mathrm{K}_0^{\mathrm{a}}\big(\mathrm{B}^{2n}\,,\,\mathrm{S}^{2n-1}\big)\ \xrightarrow{\approx}\ \mathrm{K}_1^{\mathrm{a}}\big(\,\mathrm{S}^{2n-1}\big)\ . \tag{3.21}$$

This determines an element of the compactly supported degree zero K-homology of $\mathbb{V}_{2n}$ associated to the noncommutative soliton. We have in this way provided a description of D-branes in terms of algebras $\mathcal{A}$ of "almost commuting" operators corresponding to "almost commutative" spaces which extend the hyperplane worldvolume $\mathbb{V}_{2n}$.

4 Gauge theory on $\mathbb{V}_{2n}^{\theta}$

In the previous section we have arrived at a description of noncommutative solitons as D-branes in terms of K-cycles (W,E,ζ). In the correspondence it was essential to introduce a connection on the complex Chan-Paton vector bundle $E\to W$. It is natural to now construct worldvolume field theories using these connections. In the sequel we shall therefore focus our attention on *gauge theories* on noncommutative spaces and examine what their classical solutions can teach us. Worldvolume gauge theories are the essence of the novel dynamical properties of D-branes in string theory.

4.1 Projective modules

To construct gauge theory on $\mathbb{V}_{2n}^{\theta}$ we proceed in the usual way by introducing connections on projective modules over the noncommutative algebra. The natural class of projective modules are the collections of Fock modules $\mathcal{F}^q:=\mathcal{F}\oplus\cdots\oplus\mathcal{F}$ (q times). There are also the trivial free modules of rank N given by N copies $\mathcal{H}^N$ of the algebra $\mathcal{H}:=\mathbb{V}_{2n}^{\theta}$ itself. Let us now go through some general facts concerning projective modules over $\mathbb{V}_{2n}^{\theta}$ [51, 80].

Proposition 4.1. *Any finitely generated left projective module over the Moyal space $\mathbb{V}_{2n}^{\theta}$ is of the form $\mathcal{E}_{N,q}=\mathcal{H}^N\oplus\mathcal{F}^q$ for some $N,q\in\mathbb{N}_0$.*

Remark 4.1. The integer N corresponds to the rank, or zeroth Chern number $c_0(\mathcal{E}_{N,q})$, of the module. The integer q corresponds to a *topological charge* whose interpretation depends on

dimension. For example, when $n = 1$ it is the *magnetic charge* or first Chern number $c_1(\mathcal{E}_{N,q})$, while when $n = 2$ it is the *instanton number* or second Chern number $c_2(\mathcal{E}_{N,q})$.

Corollary 4.1. *The K-theory of the Moyal $2n$-space is given by* $\mathrm{K}_0(\mathbb{V}^\theta_{2n}) = \mathbb{Z} \oplus \mathbb{Z}$ *with positive cone* $\mathrm{K}_0^+(\mathbb{V}^\theta_{2n}) = \mathbb{N} \oplus \mathbb{N}$.

Remark 4.2. Any two projective modules representing the same element of K-theory are isomorphic. The K-theory of $\mathbb{V}^\theta_{2n}$ is quite different from the (compactly supported) K-theory of the ordinary topologically trivial hyperplane $\mathbb{V}_{2n}$. It allows for non-trivial topological charges q and so resembles more closely the K-theory of the sphere S^{2n}. This feature will be responsible later on for the appearence of topologically non-trivial gauge field configurations which have no counterparts in ordinary gauge theory on $\mathbb{V}_{2n}$. However, the positive cone $\mathrm{K}_0^+(\mathbb{V}^\theta_{2n})$ is different from that of S^{2n} as one cannot have stable modules with negative charge $q < 0$ in the present case. This is due to a labelling problem, because there is no way to distinguish between the algebras corresponding to the noncommutativity parameters θ and $-\theta$ (see Remark 2.3). This property will manifest itself explicitly in the classical solutions that we shall construct and its origin will be elucidated in Section 5.5. Physically, it will imply that there is no way to produce vortices from anti-vortices on $\mathbb{V}^\theta_{2n}$ by simply changing the orientation of the hyperplane.

4.2 Yang-Mills theory

It is possible to define connections within the present class of noncommutative spaces in the usual spirit and formalism of noncommutative geometry [21]. However, the noncommutative space $\mathbb{V}^\theta_{2n}$ has enough symmetries so that a simple definition will suffice for our purposes. By a *connection* ∇ on a finitely-generated left projective $\mathbb{V}^\theta_{2n}$-module $\mathcal{E}$ we will mean a collection of anti-Hermitean $\mathbb{C}$-linear operators $\nabla_i : \mathcal{E} \to \mathcal{E}$, $i = 1, \dots, 2n$ satisfying the Leibniz rule

$$\nabla_i(f \cdot v) = \partial_i(f) \cdot v + f \cdot \nabla_i(v) \tag{4.1}$$

for all $i = 1, \dots, 2n$, $v \in \mathcal{E}$ and $f \in \mathbb{V}^\theta_{2n}$. The space of connections on $\mathcal{E}$ is denoted $\mathrm{Conn}(\mathcal{E})$.

Let us find the general form of a connection on a generic projective module as specified by Proposition 4.1. It is straightforward to show that only *trivial* gauge fields arise on the Fock module $\mathcal{F}$ [40], in accordance with the interpretation that $\mathcal{F}$ is like a single "point" on the noncommutative space (see Remark 2.2).

Proposition 4.2. *If* ∇ *is a connection on the Fock module* $\mathcal{F}$, *then*

$$\nabla_i = \Pi_m \circ \partial_i \circ \Pi_m + \alpha_i \, \mathbb{1} \ , \quad i = 1, \dots, 2n$$

for some $\alpha_i \in \mathbb{C}$ *and fixed* $m \in \mathbb{N}_0$.

Proof. From the Leibniz rule (4.1) and the inner derivation property (2.18) one has the identity

$$\nabla_i(f \cdot v) - f \cdot \nabla_i(v) = \mathrm{i} \, \sum_{j=1}^{2n} \left(\theta^{-1}\right)_{ij} \left[x^j \, , \, f\right] \cdot v$$

which may be rewritten as

$$\Big[\nabla_i - \mathrm{i} \, \sum_{j=1}^{2n} (\theta^{-1})_{ij} \, x^j \, , \, f\Big] \cdot v = 0$$

for all $f \in \mathbb{V}_{2n}^\theta$ and all $v \in \mathcal{F}$. It follows that the operator $\nabla_i - \mathrm{i}\sum_j (\theta^{-1})_{ij}\, x^j$ lives in the center of the algebra $\mathrm{End}_{\mathbb{V}_{2n}^\theta}(\mathcal{F}) \cong \mathbb{V}_{2n}^\theta$ and hence is proportional to the identity endomorphism $\mathbb{1}$ of $\mathcal{F}$. □

We can construct *non-trivial* gauge fields instead on the trivial module given by the algebra $\mathcal{H} = \mathbb{V}_{2n}^\theta$ itself. By the Leibniz rule (4.1) any connection $\nabla_i : \mathcal{H} \to \mathcal{H}$ can be written in the form

$$\nabla_i(v) = -\,\mathrm{i}\sum_{j=1}^{2n} \left(\theta^{-1}\right)_{ij} x^j \cdot v + D_i(v) \tag{4.2}$$

for $v \in \mathcal{H}$, where $D_i \in \mathrm{End}(\mathcal{H})$ is any anti-Hermitean operator on $\mathcal{H}$. We choose

$$D_i = \mathrm{i}\sum_{j=1}^{2n} \left(\theta^{-1}\right)_{ij} x^j + A_i \tag{4.3}$$

where $A_i \in \mathrm{End}_{\mathbb{V}_{2n}^\theta}(\mathcal{H})$ are anti-Hermitean $\mathbb{V}_{2n}^\theta$-linear endomorphisms. As all Moyal spaces $\mathbb{V}_{2n}^\theta$, $\theta \in \mathbb{R} \setminus \{0\}$ are Morita equivalent (in fact isomorphic), A_i can be taken to be anti-Hermitean elements of the algebra $\mathbb{V}_{2n}^\theta$ itself. Then by the inner derivation property (2.18) one has

$$\nabla_i(v) = \partial_i(v) + A_i \cdot v \ . \tag{4.4}$$

As usual, we define the *curvature* of a connection to be a measure of the deviation of the mapping $\partial_i \mapsto \nabla_i$ from being a homomorphism of the Lie algebra (2.17) of automorphisms of $\mathbb{V}_{2n}^\theta$. This gives the collection of anti-Hermitean endomorphisms $F_{ij} \in \mathrm{End}_{\mathbb{V}_{2n}^\theta}(\mathcal{H}) \cong \mathbb{V}_{2n}^\theta$, $i,j = 1,\dots,2n$ defined by

$$F_{ij} = \left[\nabla_i\,,\,\nabla_j\right] = \left[D_i\,,\,D_j\right] - \mathrm{i}\left(\theta^{-1}\right)_{ij} \mathbb{1} \ . \tag{4.5}$$

The *Yang-Mills action functional* $\mathrm{YM} : \mathrm{Conn}(\mathcal{H}) \to [0,\infty)$ is defined by

$$\mathrm{YM}(\nabla) = -\tfrac14\ \mathrm{Tr}\Big[\sum_{i,j=1}^{2n} \left(F_{ij}\right)^2\Big] = -\tfrac14\ \mathrm{Tr}\Big[\sum_{i,j=1}^{2n} \left([D_i, D_j] - \mathrm{i}\,(\theta^{-1})_{ij}\,\mathbb{1}\right)^2\Big] \ . \tag{4.6}$$

The Yang-Mills functional (4.6) is invariant under the *gauge transformations*

$$A_i \ \longmapsto\ U\,\partial_i\left(U^{-1}\right) + U\,A_i\,U^{-1}\ , \quad i = 1,\dots,2n \tag{4.7}$$

with $U \in \mathrm{U}(\mathcal{H})$, which induce the unitary transformations

$$D_i \ \longmapsto\ U\,D_i\,U^{-1}\ , \quad F_{ij} \ \longmapsto\ U\,F_{ij}\,U^{-1}\ . \tag{4.8}$$

4.3 Fluxons

We can explicitly construct all exact solutions to Yang-Mills theory on the Moyal plane $\mathbb{V}_2^\theta$, and hence for the remainder of this section we will focus on this case. It is convenient to introduce formal complex combinations of the operators (4.3) using the basic elements (2.4) to write

$$D = -\tfrac{1}{\sqrt{2\,\theta}}\,a^\dagger + \tfrac12\,(A_1 - \mathrm{i}\,A_2)\ , \quad \overline{D} = \tfrac{1}{\sqrt{2\,\theta}}\,a + \tfrac12\,(A_1 + \mathrm{i}\,A_2)\ . \tag{4.9}$$

In terms of these operators the curvature (4.5) reads

$$F := F_{12} = 2\,\mathrm{i}\,\left(\big[\,\overline{D}\,,\,D\big] - \tfrac{1}{2\,\theta}\,\mathbb{1}\right) \tag{4.10}$$

and the Yang-Mills functional (4.6) becomes

$$\mathrm{YM}\big(\nabla\big) = \mathrm{YM}\big(D\,,\,\overline{D}\,\big) := 2\ \mathrm{Tr}\ \left(\big[\,\overline{D}\,,\,D\big] - \tfrac{1}{2\,\theta}\,\mathbb{1}\right)^2\ . \tag{4.11}$$

Applying the variational principle to the action (4.11) yields the *Yang-Mills equations* on $\mathbb{V}_2^\theta$ given by

$$\big[D\,,\,\big[\,\overline{D}\,,\,D\big]\big] = \big[\,\overline{D}\,,\,\big[\,\overline{D}\,,\,D\big]\big] = 0\ . \tag{4.12}$$

As previously, we are interested in special classes of solutions to these equations.

Definition 4.1. Let $q \in \mathbb{N}$. A *q-fluxon* on $\mathbb{V}_2^\theta$ is an anti-Hermitean solution $D, \overline{D} \in \mathrm{End}(\mathcal{H})$ of the Yang-Mills equations (4.12) which has topological charge $c_1(\mathcal{H}) =\ \mathrm{Tr}\,(F) = q$ and for which the Yang-Mills functional $\mathrm{YM}(D, \overline{D}\,)$ is well-defined and finite.

A fluxon solution is a soliton on the noncommutative plane carrying a magnetic charge or "flux" [71, 40]. Using the shift operator S introduced in Example 3.3, it is straightforward to explicitly construct all finite action solutions of the Yang-Mills equations on the Moyal plane $\mathbb{V}_2^\theta$ [40].

Theorem 4.1. *To each collection* $\lambda_0, \lambda_1, \dots, \lambda_{q-1} \in \mathbb{C}$ *of fixed complex numbers there bijectively corresponds a q-fluxon*

$$D^{(q)}_{\{\lambda_i\}} = \sum_{i=0}^{q-1} \lambda_i\ e_i \otimes e_i^* - \big(\mathsf{S}\big)^q\,c^\dagger\,\big(\mathsf{S}^\dagger\big)^q\ , \quad \overline{D}\,^{(q)}_{\{\lambda_i\}} = \sum_{i=0}^{q-1} \overline{\lambda}_i\ e_i \otimes e_i^* + \big(\mathsf{S}\big)^q\,c\,\big(\mathsf{S}^\dagger\big)^q$$

in $\mathrm{End}(\mathcal{H})$ *of action*

$$\mathrm{YM}\big(D^{(q)}_{\{\lambda_i\}}\,,\,\overline{D}\,^{(q)}_{\{\lambda_i\}}\big) = 2\pi\,\theta^{-1}\,q\ ,$$

where $c, c^\dagger$ *generate the irreducible representation of the Heisenberg algebra.*

Proof. We need to find a pair of anti-Hermitean operators $D, \overline{D}$ on the free module $\mathcal{H}$ over $\mathbb{V}_2^\theta$ which obey the equations

$$\big[D\,,\,\overline{D}\,\big] = \tfrac{1}{2}\left(\tfrac{1}{\theta}\,\mathbb{1} + \mathrm{i}\,F\right)\ , \quad \big[D\,,\,F\big] = \big[\,\overline{D}\,,\,F\big] = 0\ .$$

These equations imply that $D, \overline{D}, \frac{1}{2}\left(\frac{1}{\theta} + \mathrm{i}\,F\right)$ form a representation of the Heisenberg commutation relations, with the curvature F generating the center of the algebra. Under the action of these operators, the module $\mathcal{H}$ thereby decomposes as

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n$$

into irreducible representations $\mathcal{H}_n \cong \mathcal{F}$ of this Heisenberg algebra. Then for each n one has $F|_{\mathcal{H}_n} = f_n\,\mathbb{1}$ for some $f_n \in \mathrm{i}\,\mathbb{R}$. Set $d_n := \dim(\mathcal{H}_n)$. Then d_n is infinite unless $1 + \mathrm{i}\,\theta\,f_n = 0$. The finite action constraint requires $\mathrm{Tr}\,(F^2) < \infty$, where

$$\mathrm{Tr}\,\big(F^2\big) = \sum_n d_n\,f_n^2\ .$$

This is a sum of negative terms. If d_n is infinite for some n, then the only way to make this quantity well-defined is to have $f_n = 0$ in such a way that the regulated trace yields a finite product $d_n\, f_n^2$. On the other hand, if some $d_n \in \mathbb{N}$ then $f_n = -\frac{\mathrm{i}}{\theta}$, and the finite action condition implies that there are only finitely many such positive dimensions.

These facts imply that the fluxon solution is determined by a finite-dimensional linear subspace $V_q \subset \mathcal{H}$ which may be characterized as follows. Via a gauge transformation if necessary, we may assume that V_q is the linear span of the number basis vectors $e_0, e_1, \dots, e_{q-1}$. On V_q, the operators D and $\overline{D}$ commute, $[D, \overline{D}\,] = 0$, and without loss of generality may be taken to be diagonal operators with respect to the chosen basis of V_q so that

$$D\,\big|_{V_q} = \sum_{i=0}^{q-1} \lambda_i\ e_i \otimes e_i^* \ , \quad \overline{D}\,\big|_{V_q} = \sum_{i=0}^{q-1} \overline{\lambda}_i\ e_i \otimes e_i^*$$

for some fixed $\lambda_0, \lambda_1, \dots, \lambda_{q-1} \in \mathbb{C}$. On the orthogonal complement $\mathcal{H} \ominus V_q \cong \mathcal{H}$, there exists $N \in \mathbb{N}$ such that the operators are instead generically given by a reducible sum of N irreducible representations of the Heisenberg algebra as

$$D\,\big|_{\mathcal{H}\ominus V_q} = \bigoplus_{k=0}^{N-1} \big(-c_{(k)}^\dagger\big) \ , \quad \overline{D}\,\big|_{\mathcal{H}\ominus V_q} = \bigoplus_{k=0}^{N-1} \big(c_{(k)}\big)$$

with $[c_{(k)}, c_{(k)}^\dagger] = 1$ for each $k = 0, 1, \dots, N-1$. By the Stone-von Neumann theorem, $c_{(k)} = c$ and $c_{(k)}^\dagger = c^\dagger$ for each k and hence

$$D\,\big|_{\mathcal{H}\ominus V_q} = -c^\dagger \otimes \mathbb{1}_N \ , \quad \overline{D}\,\big|_{\mathcal{H}\ominus V_q} = c \otimes \mathbb{1}_N \ .$$

This makes the Hilbert space $\mathcal{H} \ominus V_q$ into an N-fold Fock module $\mathcal{F}^N \cong \ell^2(\mathbb{N}_0^N)$ with number basis $e_n^{(k)}$, $n \in \mathbb{N}_0$, $k = 0, 1, \dots, N-1$ defined by the actions

$$\begin{aligned} c^\dagger \cdot e_n^{(k)} &= \sqrt{n+1}\ e_{n+1}^{(k)} \ , \\ c \cdot e_n^{(k)} &= \sqrt{n}\ e_{n-1}^{(k)} \ , \\ \mathbb{1}_N \cdot e_n^{(k)} &= e_n^{(k)} \ . \end{aligned}$$

Let us take $N = 1$. Let $\mathsf{S}_q{}^\dagger : \mathcal{H} \ominus V_q \to \mathcal{H}$ be a unitary isomorphism of $\mathbb{V}_2^\theta$-modules. Extend $\mathsf{S}_q{}^\dagger$ to all of $\mathcal{H}$ by setting it equal to 0 on V_q. Then as an operator on $\mathcal{H}$ it satisfies

$$\mathsf{S}_q{}^\dagger\, \mathsf{S}_q = \mathbb{1} \ , \quad \mathsf{S}_q\, \mathsf{S}_q{}^\dagger = \mathbb{1} - \mathsf{P}_q$$

where P_q is the orthogonal projection $\mathcal{H} \to V_q$. In other words, the endomorphism $\mathsf{S}_q \in \mathrm{End}(\mathcal{H})$ is a partial isometry of $\mathcal{H}$. With respect to the chosen number basis, we have $\mathsf{P}_q = \mathsf{P}_{(q)} = \sum_{i=0}^{q-1} e_i \otimes e_i^*$ and $\mathsf{S}_q = (\mathsf{S})^q$ where S is the shift endomorphism introduced in Example 3.3. The conclusion now follows from computing the corresponding curvature to get $F = \frac{\mathrm{i}}{\theta}\, \mathsf{P}_q$. □

Remark 4.3. The q-fluxon is labelled by the set of moduli $\lambda_i \in \mathbb{C}$, $i = 0, 1, \dots, q-1$ which describe the positions or separations of the *vortices* (carrying magnetic charge $q \in \mathbb{N}$) on $\mathbb{V}_2$ [40]. The explicit solution constructed above is of rank 1. Higher rank fluxons can be similarly constructed by choosing $N > 1$ in the proof of Theorem 4.1, with no qualitative change by the Hilbert hotel argument. Thus, in addition to their moduli, fluxons are labelled by K-theory charges $(N, q) \in \mathrm{K}_0^+(\mathbb{V}_2^\theta)$. Identifying the gauge equivalence classes of fluxon solutions now consists in quotienting by the discrete Weyl subgroup $S_q \subset \mathrm{U}(q) \subset \mathrm{U}(\mathcal{H})$ acting non-trivially on the subspace V_q above by permuting the fluxon positions λ_i.

Corollary 4.2. *The moduli space* $\mathcal{G}_{N,q}(\mathbb{V}_2^\theta)$ *of fluxons of K-theory charge* (N,q) *is the q-th symmetric product orbifold*

$$\mathcal{G}_{N,q}(\mathbb{V}_2^\theta) \;=\; \mathrm{Sym}^q(\mathbb{V}_2) \ .$$

Remark 4.4. The value of the Yang-Mills functional on a fluxon as in Theorem 4.1 diverges in the formal limit $\theta \to 0$, consistent with the lack of finite action topologically non-trivial field configurations in ordinary gauge theory on the plane $\mathbb{V}_2$. Note that the fluxon solution appears with only one sign of the topological charge q, consistent with the K-theory description of Section 4.1. These configurations are not global minima of the Yang-Mills functional in general and hence are *unstable*. In string theory they can be interpreted as describing q unstable D0-branes inside N D2-branes with worldvolume $\mathbb{V}_2$, in a background B-field and in the Seiberg-Witten limit [39, 40, 1].

5 Toroidal D-branes

In the sequel we will leave the setting of Moyal spaces and start looking at more complicated worldvolume geometries. As the simplest extension of our previous considerations, in this section we will still work with flat target spaces X but we will assume that the worldvolume is compactified on a two-dimensional torus T^2. This has the effect of bringing in non-trivial topological effects while still retaining the relative simplicity of flat worldvolume geometries. The presence of the constant B-field deforms the worldvolume to a noncommutative torus T^2_Θ which is the most studied and best understood example of a noncommutative space [21]. After a quick reminder of some pertinent aspects of the noncommutative geometry of this space, we will construct all (finite action) solutions of the corresponding Yang-Mills equations. We will then show that these solutions possess a remarkably intimate connection with the fluxon solutions constructed in the previous section.

5.1 Solitons on the noncommutative torus

The two-dimensional noncommutative torus T^2_Θ is the classic example of a noncommutative space and we will only briefly mention some facets of its geometry, primarily to set notation. Unless otherwise explicitly stated, we will fix an irrational number $\Theta \in (0,1) \cap (\mathbb{R} \setminus \mathbb{Q})$. We define T^2_Θ to be the associative unital $*$-algebra generated by a pair of unitaries U_1, U_2 obeying the single relation

$$U_1\,U_2 = \mathrm{e}^{\,2\pi\,\mathrm{i}\,\Theta}\ U_2\,U_1 \ . \tag{5.1}$$

As with Moyal spaces, we will regard T^2_Θ as an appropriate "algebra of Schwartz functions" defined in terms of expansions in the generators U_1, U_2. There is again a natural trace Tr : $\mathrm{T}^2_\Theta \to \mathbb{C}$ on the algebra which is positive and faithful, and hence a natural notion of integral. A collection of linear derivations $\partial_i : \mathrm{T}^2_\Theta \to \mathrm{T}^2_\Theta$ may be defined on generators by

$$\partial_i(U_j) = 2\pi\,\mathrm{i}\,\delta_{ij}\ U_i \ , \quad i,j = 1,2 \tag{5.2}$$

and extended as automorphisms of T^2_Θ by linearity and the Leibniz rule. As in the Moyal case these derivations commute,

$$\big[\partial_i \,,\, \partial_j\big] = 0 \ , \tag{5.3}$$

and they generate the action of the two-dimensional translation group of T^2 on the noncommutative algebra. There is also a Weyl-Wigner correspondence completely analogous to that of Section 2.3 which may be used to define T^2_Θ via deformation quantization [86], but we will not need this formalism here.

It is possible to now proceed in an analogous way to Section 3 with the construction of scalar field theories on T^2_Θ and their soliton solutions [62, 10, 50, 53]. The resulting configurations are similar to those of Moyal spaces, except that the non-trivial topology now leads to far richer structures. Explicit projector solitons are provided by the Powers-Rieffel projector on T^2_Θ [75]. Its Wigner function is not a localized solitonic "lump" like the Gaussian soliton field configurations on $\mathbb{V}^\theta_{2n}$. Rather, it is localized along one direction but yields stripe-like patterns in the other direction of T^2 [10, 55]. On the other hand, the homotopically equivalent Boca projector [13] more closely resembles the Gaussian projector solitons on $\mathbb{V}^\theta_{2n}$ [53, 55]. Partial isometry solitons on T^2_Θ can also be constructed as the "angular" operators appearing in the polar decompositions of closed bounded operators based on the Powers-Rieffel projection [55], in analogy to the angular dependence of the Wigner functions for the basic shift partial isometry S on $\mathbb{V}^\theta_{2n}$ (see Example 2.1). The explicit descriptions of these configurations are much more involved than in the Moyal case and are beyond the scope of these notes. Instead, we will now proceed to focus our attention to gauge theories on the noncommutative torus.

5.2 Gauge theory

We begin by describing the K-theory of the noncommutative torus [21].

Proposition 5.1. *The K-theory of the noncommutative torus is given by the ordered subgroup* $\mathrm{K}_0(\mathrm{T}^2_\Theta) = \mathbb{Z} + \mathbb{Z}\,\Theta$ *of* $\mathbb{R}$ *with positive cone* $\mathrm{K}_0^+(\mathrm{T}^2_\Theta) = \{(p,q) \in \mathbb{Z}^2 \mid p - q\,\Theta > 0\}$.

Proof. K-theory is stable under deformations of unital algebras so $\mathrm{K}_0(\mathrm{T}^2_\Theta) \cong \mathrm{K}_0(C(\mathrm{T}^2)) = \mathbb{Z}\oplus\mathbb{Z}$. For $(p,q) \in \mathbb{Z}^2$, let $\mathsf{P} = \mathsf{P}_{p,q} \in \mathbb{M}_n(\mathrm{T}^2_\Theta)$, $n \in \mathbb{N}$ be a projector representing the corresponding Murray-von Neumann equivalence class. One has

$$\mathrm{Tr}\,\big(\mathsf{P}_{p,q}\big) = \ \mathrm{Tr}\,\big(\mathsf{P}_{p,q}\,{\mathsf{P}_{p,q}}^\dagger\big) = p - q\,\Theta$$

and hence the trace yields an isomorphism $\mathrm{Tr}\ : \mathrm{K}_0(\mathrm{T}^2_\Theta) \xrightarrow{\approx} \mathbb{Z} + \mathbb{Z}\,\Theta$, with the trivial identity projector $\mathbb{1}$ generating the first copy of $\mathbb{Z}$ and the Powers-Rieffel projector generating the second copy of $\mathbb{Z}$. The trace of $\mathsf{P}_{p,q}$ coincides with the Murray-von Neumann dimension of its image subspace. □

For $(p,q) \in \mathbb{Z}^2$, the space $\mathcal{E}_{p,q} = \mathsf{P}_{p,q}(\mathrm{T}^2_\Theta)^n$ is a finitely generated left projective module over T^2_Θ called a *Heisenberg module.*

Proposition 5.2. *Any finitely generated projective module over the noncommutative torus* T^2_Θ *which is not free is isomorphic to a Heisenberg module* $\mathcal{E}_{p,q}$.

Henceforth we will consider only Heisenberg modules over the noncommutative torus. The module $\mathcal{E}_{p,q}$ is a stable module labelled by the positive cone $(p,q) \in \mathrm{K}_0^+(\mathrm{T}^2_\Theta)$ of the K-theory group and has positive Murray-von Neumann dimension

$$\dim(\mathcal{E}_{p,q}) = \mathrm{Tr}\,(\mathsf{P}_{p,q}) = p - q\,\Theta > 0 \ . \tag{5.4}$$

The integer

$$q = \tfrac{1}{2\pi\,\mathrm{i}}\ \mathrm{Tr}\,\big[\mathsf{P}_{p,q}\,\big(\partial_1(\mathsf{P}_{p,q})\,\partial_2(\mathsf{P}_{p,q}) - \partial_2(\mathsf{P}_{p,q})\,\partial_1(\mathsf{P}_{p,q})\big)\big] \tag{5.5}$$

is the Chern number $c_1(\mathcal{E}_{p,q})$ and will be referred to as the *topological charge* of the Heisenberg module. We define the *rank* N of $\mathcal{E}_{p,q}$ to be the positive integer

$$N = \gcd(p,q) \ . \tag{5.6}$$

We will always assume $q > 0$ in what follows.

We define connections ∇ on Heisenberg modules analogously to the Moyal case, i.e. as pairs of anti-Hermitean $\mathbb{C}$-linear operators $\nabla_1, \nabla_2 : \mathcal{E}_{p,q} \to \mathcal{E}_{p,q}$ obeying the usual Leibniz rule analogously to eq. (4.1). The space of connections on a Heisenberg module is denoted $\mathrm{Conn}(\mathcal{E}_{p,q})$. We can use the automorphisms ∂_i given by eq. (5.2) to define fixed fiducial connections and write $\nabla_i = \mathsf{P}_{p,q} \circ \partial_i \circ \mathsf{P}_{p,q} + A_i$ with $A_i \in \mathrm{End}_{\mathrm{T}^2_\Theta}(\mathcal{E}_{p,q})$. The curvature $F \in \mathrm{End}_{\mathrm{T}^2_\Theta}(\mathcal{E}_{p,q})$ of a connection ∇ is defined as usual by

$$F := \big[\nabla_1\,,\,\nabla_2\big] \ , \tag{5.7}$$

and one may introduce the Yang-Mills functional $\mathrm{YM} : \mathrm{Conn}(\mathcal{E}_{p,q}) \to [0,\infty)$ on a *fixed* Heisenberg module as

$$\mathrm{YM}(\nabla) = -\tfrac{\tau}{2}\ \mathrm{Tr}\,\big([\nabla_1,\nabla_2]^2\big) = -\tfrac{\tau}{2}\ \mathrm{Tr}\,\big(F^2\big) \ . \tag{5.8}$$

For later convenience we have introduced a constant parameter $\tau > 0$ which we identify as the (imaginary part of the) modulus of the torus T^2. It should accompany all trace operations in order to keep quantities consistently defined. The Yang-Mills functional (5.8) is invariant under the gauge transformations

$$\nabla_i \ \longmapsto\ U\,\nabla_i\,U^{-1} \ , \quad i = 1,2 \tag{5.9}$$

with $U \in \mathrm{U}(\mathcal{E}_{p,q})$.

Proposition 5.3. *Any Heisenberg module $\mathcal{E}_{p,q}$ admits a connection ∇^{c} of constant curvature*

$$F_{p,q} := \big[\nabla_1^{\mathrm{c}}\,,\,\nabla_2^{\mathrm{c}}\big] = \frac{2\pi\,\mathrm{i}}{\tau}\,\frac{q}{p - q\,\Theta}\ \mathsf{P}_{p,q} \ .$$

Remark 5.1. Since $\mathcal{E}_{p,q} = \mathsf{P}_{p,q}(\mathrm{T}^2_\Theta)^n$ for some $n \in \mathbb{N}$ and $(\mathsf{P}_{p,q})^2 = \mathsf{P}_{p,q}$, the projector $\mathsf{P}_{p,q}$ acts as the identity endomorphism on the Heisenberg module and hence the curvature $F_{p,q}$ is "constant". It now follows that the topological charge (5.5) can be expressed in the standard form

$$q = \tfrac{\tau}{2\pi\,\mathrm{i}}\ \mathrm{Tr}\,(F_{p,q}) \ .$$

The equality of the two expressions for the topological charge is essentially the index theorem.

Using Proposition 5.3 we can obtain an explicit description of the Heisenberg module $\mathcal{E}_{p,q}$ as the separable Hilbert space

$$\mathcal{E}_{p,q} = \mathcal{F} \otimes \mathcal{W}_{p,q} \ . \tag{5.10}$$

The Fock module $\mathcal{F}$ is the irreducible representation of the Heisenberg commutation relation $[\nabla_1^{\rm c}, \nabla_2^{\rm c}] = F_{p,q}$, while $\mathcal{W}_{p,q} \cong \mathbb{C}^q$ is the $q \times q$ representation of the Weyl algebra

$$\Gamma_1\,\Gamma_2 = {\rm e}^{\,2\pi\,{\rm i}\,p/q}\ \Gamma_2\,\Gamma_1 \tag{5.11}$$

whose unique irreducible module has rank $\frac{q}{\gcd(p,q)} = \frac{q}{N}$. The generators U_1, U_2 of the noncommutative torus then act on $\mathcal{E}_{p,q}$ as the operators

$$U_i = \exp\big(\tfrac{\sqrt{\tau}}{q}\,(p - q\,\Theta)\,\nabla_i^{\rm c}\big) \otimes \Gamma_i \ , \quad i = 1, 2 \ . \tag{5.12}$$

5.3 Instantons

Varying the Yang-Mills functional (5.8) in the usual way leads to the Yang-Mills equations

$$\big[\nabla_1\,,\,\big[\nabla_1\,,\,\nabla_2\big]\big] = \big[\nabla_2\,,\,\big[\nabla_1\,,\,\nabla_2\big]\big] = 0 \tag{5.13}$$

on the Heisenberg module $\mathcal{E}_{p,q}$. As always, we are interested in the finite action solutions to these equations up to gauge equivalence defined by the transformations (5.9).

Definition 5.1. An *instanton* of K-theory charge $(p,q) \in {\rm K}_0^+({\rm T}_\Theta^2)$ on ${\rm T}_\Theta^2$ is a solution $\nabla \in {\rm Conn}(\mathcal{E}_{p,q})$ of the Yang-Mills equations (5.13) on the Heisenberg module $\mathcal{E}_{p,q}$ for which the Yang-Mills functional ${\rm YM}(\nabla)$ is well-defined and finite.

The explicit construction of such solutions is more involved than on the Moyal plane $\mathbb{V}_2^\theta$, because in the present case the derivations (5.2) are *outer* automorphisms of the noncommutative algebra.

An obvious solution to the Yang-Mills equations (5.13) is the constant curvature connection $\nabla = \nabla^{\rm c}$ on $\mathcal{E}_{p,q}$ of topological charge q. It has action

$${\rm YM}(\nabla^{\rm c}) = \frac{4\pi^2}{\tau}\,\frac{q^2}{p - q\,\Theta} \ , \tag{5.14}$$

and one can show that this is the absolute minimum value of the Yang-Mills functional on ${\rm Conn}(\mathcal{E}_{p,q})$ [22].

Proposition 5.4. ${\rm YM}(\nabla^{\rm c}) = \inf\limits_{\nabla \in {\rm Conn}(\mathcal{E}_{p,q})} {\rm YM}(\nabla)$.

Remark 5.2. Proposition 5.4 implies that the constant curvature connections define *stable* vacuum states in noncommutative gauge theory. In the string theory setting, they correspond to $\frac{1}{2}$-BPS configurations [51].

The remaining instanton solutions to eqs. (5.13) are *unstable* and may be completely classified as follows [76, 68].

Definition 5.2. A *partition* of the K-theory charge $(p,q) \in {\rm K}_0^+({\rm T}_\Theta^2)$ is a collection $\underline{(p,q)} = \{(p_k, q_k)\}$ of charges $(p_k, q_k) \in {\rm K}_0^+({\rm T}_\Theta^2)$ for which

$$(p,q) = \sum_k\,(p_k, q_k) \ .$$

Theorem 5.1. *To each partition* $\underline{(p,q)} = \{(p_k,q_k)\}$ *of* $(p,q) \in \mathrm{K}_0^+(\mathrm{T}^2_\Theta)$ *with finitely many components* (p_k,q_k) *there bijectively corresponds an instanton* $\nabla = \nabla_{\underline{(p,q)}}$ *of K-theory charge* (p,q) *with action*

$$\mathrm{YM}\big(\nabla_{\underline{(p,q)}}\big) = \frac{4\pi^2}{\tau} \sum_k \frac{q_k^2}{p_k - q_k\,\Theta} \ .$$

Proof. The idea of the proof is similar to that of Theorem 4.1. An instanton ∇ satisfies the equations

$$\big[\nabla_1\,,\,\nabla_2\big] = F \ , \quad \big[\nabla_1\,,\,F\big] = \big[\nabla_2\,,\,F\big] = 0$$

which describes a Heisenberg algebra with generators ∇_1, ∇_2 and F giving the central element. Under the action of these operators, the Heisenberg module $\mathcal{E}_{p,q}$ thus decomposes into irreducible representations of this algebra to give

$$\mathcal{E}_{p,q} = \bigoplus_k \mathcal{E}_{p_k,q_k}$$

for some $(p_k,q_k) \in \mathbb{Z}^2$, with each submodule $\mathcal{E}_{p_k,q_k}$ a Heisenberg module over T^2_Θ, i.e. $p_k - q_k\,\Theta > 0$. By the Yang-Mills equations, a solution $\nabla : \mathcal{E}_{p_k,q_k} \to \mathcal{E}_{p_k,q_k}$ preserves the subspaces $\mathcal{E}_{p_k,q_k}$. As a consequence, $\nabla^{\mathrm{c}}_{(k)} := \nabla|_{\mathcal{E}_{p_k,q_k}}$ has constant curvature $F_{p_k,q_k} = F|_{\mathcal{E}_{p_k,q_k}}$. The Yang-Mills functional is additive,

$$\mathrm{YM}\big(\bigoplus_k \nabla_{(k)}\big) = \sum_k \mathrm{YM}\big(\nabla_{(k)}\big) \ ,$$

and by Proposition 5.4 the constant curvature connection $\nabla_{(k)} = \nabla^{\mathrm{c}}_{(k)}$ is the global minimum of $\mathrm{YM}\big|_{\mathcal{E}_{p_k,q_k}}$ for each k.

Thus a critical point of the Yang-Mills functional on this direct sum decomposition is given by

$$\nabla = \nabla_{\underline{(p,q)}} := \bigoplus_k \nabla^{\mathrm{c}}_{(k)} \ ,$$

and every classical solution of noncommutative Yang-Mills theory is characterized by module splittings in this way. The value of $\mathrm{YM}\big(\nabla_{\underline{(p,q)}}\big) = \sum_k \mathrm{YM}(\nabla^{\mathrm{c}}_{(k)})$ is given from eq. (5.14). Since each term in this sum is positive, the sum is finite only if there are finitely many terms in the decomposition. Since the module decomposition is an orthogonal direct sum, one has $\dim(\mathcal{E}_{p,q}) = \sum_k \dim(\mathcal{E}_{p_k,q_k})$ which is equivalent to

$$p - q\,\Theta = \sum_k (p_k - q_k\,\Theta) \ .$$

Since Θ is an irrational number, this is equivalent to the partition conditions. □

Remark 5.3. The conservation of topological charge

$$q = \sum_k q_k$$

arising from the partition condition characterizes Yang-Mills theory on a fixed Heisenberg module $\mathcal{E}_{p,q}$. If we extrapolate these solutions to the commutative case $\Theta = 0$ of ordinary gauge theory on T^2, then this theory is distinguished from "physical" Yang-Mills theory which would sum over all topological charges [68, 69]. In the noncommutative case only the fixed module definition

of Yang-Mills theory is well-defined, as there is no way in this case to naturally "separate" the topological numbers p and q. Note that the global minimum of the Yang-Mills functional corresponds to the special case of the trivial partition $\underline{(p,q)} = \{(p,q)\}$ having only a single component.

Remark 5.4. Since $\underline{(p,q)}$ contains only finitely many components (although it can have an arbitrarily large number), one can pick out from the corresponding module decomposition the submodule of minimum dimension and thus order the partition components according to increasing Murray-von Neumann dimension [68]. The set of values of the Yang-Mills action on its critical point set (the set of all partitions or instantons) is discrete, and thus YM defines a Morse functional on the space of connections $\mathrm{Conn}(\mathcal{E}_{p,q})$ [76].

5.4 Instanton moduli spaces

It is straightforward to construct the moduli spaces of the solutions obtained in Section 5.3. We first consider the moduli space $\mathcal{G}_{(p,q)}(\mathrm{T}^2_\Theta)$ of constant curvature connections $\nabla^{\rm c}$ on $\mathcal{E}_{p,q}$ corresponding to the global minima of the Yang-Mills functional [22, 68].

Theorem 5.2. *The moduli space $\mathcal{G}_{(p,q)}(\mathrm{T}^2_\Theta)$ of stable instantons on the Heisenberg module $\mathcal{E}_{p,q}$ is the N-th symmetric product orbifold*

$$\mathcal{G}_{(p,q)}\big(\mathrm{T}^2_\Theta\big) ~=~ \mathrm{Sym}^N\big(\tilde{\mathrm{T}}^2\big)$$

of the dual torus $\tilde{\mathrm{T}}^2$, where $N=\gcd(p,q)$ is the rank of $\mathcal{E}_{p,q}$.

Proof. Write the realization (5.10) of the Heisenberg module as

$$\mathcal{E}_{p,q}=\mathcal{F}\otimes\big(\mathcal{W}_{\zeta_1}\oplus\cdots\oplus\mathcal{W}_{\zeta_N}\big)$$

where, for each $i=1,\dots,N$, the module $\mathcal{W}_{\zeta_i}\cong\mathbb{C}^{q/N}$ is the irreducible representation of the Weyl algebra (5.11) with central generator $\zeta_i\in\tilde{\mathrm{T}}^2$. The only gauge transformations (5.9) which act non-trivially on this decomposition live in the Weyl subgroup $S_N\subset\mathrm{U}(N)\subset\mathrm{U}(\mathcal{E}_{p,q})$ and act by permuting the various components $\mathcal{W}_{\zeta_i}$ of the direct sum. Dividing by this subgroup gives the desired moduli space. □

Remark 5.5. As in the case of fluxons, the stable instanton moduli space coincides with the quantum mechanical configuration space of a number of identical particles on $\tilde{\mathrm{T}}^2$. In this sense the gauge theory is "topological" [68, 70], in that it resembles more closely quantum mechanics rather than field theory. Notice that the number of instantons here is determined by the rank N of the gauge theory, unlike the fluxon number which is determined by the topological charge q (Corollary 4.2). In Section 5.5 we shall explain the connection between these two classes of gauge theory solitons. The noncommutative instanton moduli space coincides with the moduli space

$$\mathcal{G}_{(p,q)}\big(\mathrm{T}^2_\Theta\big)\cong\mathrm{Hom}\big(\pi_1(\mathrm{T}^2)\,,\,\mathrm{U}(N)\big)\,/\,\mathrm{U}(N)$$

of *flat* principal $\mathrm{U}(N)$-bundles over the torus which arises in ordinary gauge theory on T^2 [9]. In the noncommutative setting, the moduli space is completely determined by the symmetric product not only for flat connections but also for *all* constant curvature connections. In the commutative case, a constant curvature connection on a principal $\mathrm{U}(N)$-bundle over T^2 can be

described as a flat connection on a non-trivial principal bundle over T^2 with structure group $\mathrm{U}(N)\,/\,\mathrm{U}(1) \cong \mathrm{SU}(N)\,/\,\mathbb{Z}_N$.

For the general case one has to be more careful because the distinct unstable instantons that we have constructed in Section 5.3 do not represent gauge equivalence classes [68]. From (5.10) and the reducibility of the corresponding Weyl algebra representation there is an isomorphism $\mathcal{E}_{m\,p,m\,q} \cong \oplus^m\,\mathcal{E}_{p,q}$ for any $m \in \mathbb{N}$. Both Heisenberg modules $\mathcal{E}_{p,q}$ and $\mathcal{E}_{m\,p,m\,q}$ have the *same* constant curvature $F_{p,q} = F_{m\,p,m\,q}$ and hence the corresponding instanton solutions should be identified through gauge invariance. This difficulty can be circumvented by modifying our previous definition of partition [68, 76].

Definition 5.3. An *instanton partition* of the K-theory charge $(p,q) \in \mathrm{K}_0^+(\mathrm{T}^2_\Theta)$ is a collection $\underline{\underline{(p,q)}} = \{(N_k, p_k, q_k)\}$ of triples of integers such that

1. $\underline{(p,q)} = \{(N_k\,p_k, N_k\,q_k)\}$ is a partition of (p,q);
2. $(p_k, q_k) \neq (p_l, q_l)$ for $k \neq l$; and
3. p_k and q_k are relatively prime for each k.

Given an arbitrary partition $\underline{(p,q)} = \{(p_k, q_k)\}$ of $(p,q) \in \mathrm{K}_0^+(\mathrm{T}^2_\Theta)$, we may write $(p_k, q_k) = N_k\,(p'_k, q'_k)$ for each k with $N_k := \gcd(p_k, q_k)$ and p'_k, q'_k coprime. Definition 5.3 then restricts to those partitions with *distinct* K-theory charges (p'_k, q'_k). They represent the distinct gauge equivalence classes of instanton solutions to the Yang-Mills equations [68, 76].

Theorem 5.3. *The moduli space* $\mathcal{G}_{\underline{\underline{(p,q)}}}(\mathrm{T}^2_\Theta)$ *of instantons on the Heisenberg module* $\mathcal{E}_{p,q}$ *corresponding to an instanton partition* $\underline{\underline{(p,q)}} = \{(N_k, p_k, q_k)\}$ *is given by*

$$\mathcal{G}_{\underline{\underline{(p,q)}}}(\mathrm{T}^2_\Theta) \;=\; \prod_k \mathrm{Sym}^{N_k}\big(\tilde{\mathrm{T}}^2\big) \ .$$

Proof. The partition $\{(N_k\,p_k, N_k\,q_k)\}$ modifies the module decompositions used in the proof of Theorem 5.1 to

$$\mathcal{E}_{p,q} = \bigoplus_k \mathcal{E}_{N_k\,p_k, N_k\,q_k}$$

where the constant curvatures of the submodules $\mathcal{E}_{N_k\,p_k, N_k\,q_k}$ are all distinct. Since gauge transformations preserve the constant curvature conditions, they also preserve each submodule $\mathcal{E}_{N_k\,p_k, N_k\,q_k}$ and so the instanton moduli space is given by the product

$$\mathcal{G}_{\underline{\underline{(p,q)}}}(\mathrm{T}^2_\Theta) = \prod_k \mathcal{G}_{(N_k\,p_k, N_k\,q_k)}(\mathrm{T}^2_\Theta) \ ,$$

where $\mathcal{G}_{(N_k\,p_k, N_k\,q_k)}(\mathrm{T}^2_\Theta)$ is the moduli space of constant curvature connections on the Heisenberg module $\mathcal{E}_{N_k\,p_k, N_k\,q_k}$. The result now follows by Theorem 5.2. $\square$

5.5 Decompactification

The classification of classical solutions which we have given is completely analogous to that which arises in *ordinary* gauge theory on a two-dimensional torus T^2. In the commutative case it is just the Atiyah-Bott bundle-splitting construction [9, 37, 68] and can be obtained in the present case by formally setting $\Theta = 0$ everywhere. The integers p and p_k then represent the true ranks of Hermitean vector bundles over T^2. The difference in the noncommutative case lies in the structure of the partitions that can contribute to finite action solutions [68]. Furthermore, in the commutative case if we were to *decompactify* the torus T^2 onto the plane $\mathbb{V}_2$ by formally sending the modulus $\tau \to \infty$, then the instanton solutions would "disappear", in the sense that their action vanishes in this limit. This follows by setting $\Theta = 0$ in Theorem 5.1, and it is consistent with the fact that there are no topologically non-trivial finite action gauge field configurations on the plane. Instead, as we will now show, the formal "decompactification" of the noncommutative torus T^2_Θ onto the Moyal plane $\mathbb{V}_2^\theta$ maps the instantons of this section in a very precise way onto the fluxons constructed in Section 4.3 [38]. This provides a rather remarkable limiting construction of the intricate fluxon solutions given by Theorem 4.1 in terms of the relatively simpler noncommutative torus instantons obtained through the Fock module realizations provided by Proposition 5.3.

We introduce the quantity

$$\theta := \frac{\tau\,\Theta}{2\pi} \tag{5.15}$$

which will turn out to be the noncommutativity parameter of the Moyal plane $\mathbb{V}_2^\theta$. We take the limits $\tau \to \infty$, $\Theta \to 0$ whilst keeping the combination (5.15) fixed. From the form of the Yang-Mills functional given by Theorem 5.1, the only finite action configurations which survive this limit are those partitions $\underline{(p,q)} = \underline{(0,-q)}_{\,0} := \{(0,-q_k)\}$ having $p_k = 0$ and $q_k, q > 0$ for all k (so that $q_k\,\Theta, q\,\Theta > 0$ by the partition constraints). The action of these instantons may be written in terms of the parameter (5.15) as

$$\mathrm{YM}\big(\nabla_{\underline{(0,-q)}_{\,0}}\big) = \frac{2\pi\,q}{\theta}~. \tag{5.16}$$

By Theorem 4.1 this is the action of a fluxon of topological charge $q \in \mathbb{N}$.

To provide the explicit mapping of instantons onto fluxons, let us fix a finite action decompactification partition $\underline{(0,-q)}_{\,0}$. For $p = 0$ the Weyl algebra reads $\Gamma_1\,\Gamma_2 = \Gamma_2\,\Gamma_1$. With respect to the canonical orthonormal basis w_k, $k = 0,1,\dots,q-1$ of $\mathcal{W}_{0,-q}$, this equation is generically solved up to unitary isomorphism by operators of the form

$$\Gamma_i = \exp\big(-\tfrac{2\pi\,\mathrm{i}}{\sqrt{\tau}}\,\boldsymbol{\lambda}^i\big) := \sum_{k=0}^{q-1}\,\mathrm{e}^{\,-\frac{2\pi\,\mathrm{i}}{\sqrt{\tau}}\,\lambda_k^i}~w_k \otimes w_k^* \tag{5.17}$$

with $\lambda_k^i \in \mathbb{R}$ for $i = 1,2$ and $k = 0,1,\dots,q-1$. The corresponding module splitting provided by Theorem 5.1 thereby yields an isomorphism

$$\mathcal{E}_{0,-q} = \bigoplus_{k=0}^{q-1}\,\mathcal{E}_{0,-q_k} \cong \mathcal{F}^q~. \tag{5.18}$$

The global minimum of the Yang-Mills functional on (5.18) is the connection with constant curvature

$$F_{0,-q} = \tfrac{\mathrm{i}}{\theta}\, \mathsf{P}_{0,-q} \ . \tag{5.19}$$

The projector $\mathsf{P}_{0,-q}$ has rank $\mathrm{Tr}\,(\mathsf{P}_{0,-q}) = q\,\Theta$, so that $\frac{1}{\Theta}\,\mathsf{P}_{0,-q}$ has integer rank $q \in \mathbb{N}$. By representing the corresponding K-theory class by the Boca projection one can show that the image of $\frac{1}{\Theta}\,\mathsf{P}_{0,-q}$ is a q-dimensional subspace of the Fock module (5.18) in the decompactification limit, and thus the curvature (5.19) coincides with that of the q-fluxon solution of Theorem 4.1 [53, 55].

Proposition 5.5. *As endomorphisms of the Fock module $\mathcal{F}$ one has*

$$\lim_{\Theta\to 0}\ \tfrac{1}{\Theta}\,\mathsf{P}_{0,-q} = \mathsf{P}_{(q)} = \sum_{k=0}^{q-1} e_k \otimes e_k^* \ .$$

In the decompactification limit, we would like to map the noncommutative torus algebra T^2_Θ onto the Moyal algebra $\mathbb{V}^\theta_2$ at the level of their representations as endomorphisms of the Fock module. At the level of generators this means that we would like to roughly identify U_i, acting on (5.18) via (5.12), with the exponential operators $\exp\big(\frac{2\pi\,\mathrm{i}}{\sqrt{\tau}}\,x^i\big)$ where the Moyal plane generators x^i obey $[x^1,x^2] = \mathrm{i}\,\theta\,\mathbb{1}$ via the Baker-Campbell-Hausdorff formula. The immediate problem which arises is that the x^i are defined on the trivial rank 1 module $\mathcal{H} = \mathbb{V}^\theta_2$, while

$$U_i = \exp\big(-\tfrac{2\pi}{\sqrt{\tau}}\,(\theta\,\nabla^{\mathrm{c}}_i \otimes \mathbb{1}_q + \mathrm{i}\,\mathbb{1} \otimes \boldsymbol{\lambda}^i)\big) \tag{5.20}$$

are defined on the Fock module $\mathcal{F}^q$. In order to make an identification of this type we need to ensure that all operators are defined on a common domain.

We first embed all the pertinent operators naturally into the projective $\mathbb{V}^\theta_2$-module $\mathcal{E}_{0,-q} \oplus \mathcal{H}$ via the definitions

$$\begin{aligned} \hat{\nabla}^{\mathrm{c}}_i &= \big(\nabla^{\mathrm{c}}_i \otimes \mathbb{1}_q\big) \ \oplus\ 0 \ , \\ \hat{\boldsymbol{\lambda}}^i &= \big(\mathbb{1} \otimes \boldsymbol{\lambda}^i\big) \ \oplus\ 0 \ , \\ \hat{x}^i &= \mathbf{0}_q \ \oplus\ x^i \ . \end{aligned} \tag{5.21}$$

Then we represent these endomorphisms on the free module $\mathcal{H}$ by finding their images under a natural unitary isomorphism of separable Hilbert spaces

$$\Xi_q\ :\ \mathcal{E}_{0,-q} \oplus \mathcal{H} \ \xrightarrow{\approx}\ \mathcal{H} \ . \tag{5.22}$$

To construct this isomorphism, let S be the shift endomorphism of the Fock module with $\ker(\mathsf{S})^q = \{0\}$ and $\ker(\mathsf{S}^\dagger)^q = \mathrm{im}(\mathsf{P}_{(q)}) \cong \mathbb{C}^q$. Then the submodule $(\mathsf{S})^q \cdot \mathcal{H}$ is the orthogonal complement in $\mathcal{H}$ of $\mathcal{E}_{0,-q} \cong \mathsf{P}_{(q)} \cdot \mathcal{H}$. The isomorphism (5.22) is therefore given explicitly by

$$\Xi_q\Big(\sum_{k=0}^{q-1} f \cdot e_k \ \oplus\ f'\Big) := \mathsf{P}_{(q)} \cdot f + (\mathsf{S})^q \cdot f' \tag{5.23}$$

for $f, f' \in \mathcal{H}$. The inverse map is given for $f \in \mathcal{H}$ by

$$\Xi_q^{-1}(f) = \sum_{k=0}^{q-1} f \cdot e_k \ \oplus\ \big(\mathsf{S}^\dagger\big)^q \cdot f \ . \tag{5.24}$$

It is straightforward to work out the action of the isomorphism (5.22) on the operators (5.21) and as endomorphisms of $\mathcal{H}$ one finds

$$\begin{aligned} \Xi_q \, \hat{x}^i \, \Xi_q^{-1} &= (\mathsf{S})^q \, x^i \, (\mathsf{S}^\dagger)^q \ , \\ \Xi_q \, \hat{\boldsymbol{\lambda}}^i \, \Xi_q^{-1} &= \sum_{k=0}^{q-1} \lambda_k^i \ e_k \otimes e_k^* \ . \end{aligned} \tag{5.25}$$

The desired identifications are now given by

$$U_i = \Xi_q \ \exp\bigl(-\tfrac{2\pi}{\sqrt{\tau}} \, (\theta \, \hat{\nabla}_i^{\rm c} + {\rm i} \, \hat{\boldsymbol{\lambda}}^i)\bigr) \, \Xi_q^{-1} := \Xi_q \ \exp\bigl(\tfrac{2\pi \, {\rm i}}{\sqrt{\tau}} \, \hat{x}^i\bigr) \, \Xi_q^{-1} \tag{5.26}$$

for $i = 1, 2$. From (5.25) it then follows finally that the decompactification limit of the constant curvature connection of Yang-Mills theory on T^2_Θ thus leads to the operators

$$D_i := {\rm i} \, \theta \, \Xi_q \, \hat{\nabla}_i^{\rm c} \, \Xi_q^{-1} = \sum_{k=0}^{q-1} \lambda_k^i \ e_k \otimes e_k^* + (\mathsf{S})^q \, x^i \, (\mathsf{S}^\dagger)^q \ . \tag{5.27}$$

This coincides with the q-fluxon given by Theorem 4.1. Thus the decompactification of finite action instantons on the noncommutative torus provides a natural and clear way to describe the fluxon solutions of Yang-Mills gauge theory on $\mathbb{V}_2^\theta$. The construction of these solutions is very natural in the toroidal framework, and the torus instanton origin of fluxons explains many of their seemingly unusual properties in precise geometric ways [38]. For instance, the constraint $q > 0$ on the sign of the fluxon charges can be traced back to the required positivity of the Murray-von Neumann dimension on the positive K-theory cone of stable Heisenberg modules, while the instability of fluxons is due to the instability of their instanton ancestors.

6 D-branes in group manifolds

In this final section we leave the setting of flat target spaces X and study some examples of D-branes in curved backgrounds. A particularly tractable class of examples is provided by the cases where $X = \mathrm{G}$ is a *group* manifold. While these spacetimes are not entirely realistic string backgrounds, they provide important solvable models which sometimes form subspaces of genuine target spaces. They possess enough symmetries so that a relatively complete classification of D-branes, and their noncommutative worldvolume geometries, may be readily obtained. We will describe the general quantization scheme for special classes of D-branes in these backgrounds, and how to construct the corresponding noncommutative worldvolume gauge theories. We then work out the simplest example of D-branes whose worldvolumes are fuzzy two-spheres in the group manifold of $\mathrm{G} = \mathrm{SU}(2)$ where everything can be made very explicit. A detailed review of these matters along with an exhaustive list of references can be found in [79].

6.1 Symmetric D-branes

Let G be a compact, simple, simply connected and connected Lie group possessing a bi-invariant metric. Let $\mathfrak{g}$ be the Lie algebra of G. The Lie bracket on $\mathfrak{g}$ is denoted $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, and the metric on G induces an invariant inner product $\langle -,- \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. As in Section 1, a string

in the target space $X = \mathrm{G}$ is a harmonic map $g : \Sigma \to \mathrm{G}$ on an oriented Riemann surface Σ. The special feature of group manifolds is that the string theory possesses an affine $\mathrm{G} \times \overline{\mathrm{G}}$ symmetry

$$g(z,\bar z) \longmapsto \Omega(z)\, g(z,\bar z)\, \overline{\Omega}(\bar z)^{-1} , \tag{6.1}$$

where $(z, \bar z)$ are local complex coordinates on Σ and $\Omega, \overline{\Omega} : \Sigma \to \mathrm{G}$ are independent holomorphic and antiholomorphic maps.

Select a fixed boundary component of $\partial\Sigma$, and choose the local parametrization of the worldsheet Σ such that this component is located at $z = \bar z$. Then the symmetry (6.1) restricts to this boundary component as

$$g\big|_{\partial\Sigma} \longmapsto \Omega\, g\big|_{\partial\Sigma}\, \overline{\Omega}^{-1} . \tag{6.2}$$

A D-brane is now a boundary condition $g\big|_{\partial\Sigma} : \partial\Sigma \to W \subset \mathrm{G}$ which preserves enough of the affine $\mathrm{G}\times\overline{\mathrm{G}}$ symmetry as dictated by conformal invariance of the underlying boundary conformal field theory.

Definition 6.1. A *symmetric D-brane* in G is a boundary condition with

$$\overline{\Omega} = \omega \circ \Omega$$

on $\partial\Sigma$ for some isometric automorphism $\omega : \mathrm{G} \to \mathrm{G}$.

Symmetric D-branes preserve a maximal diagonal subgroup $\mathrm{G} \subset \mathrm{G} \times \overline{\mathrm{G}}$ of the affine symmetry. Their worldvolumes W are *twisted conjugacy classes*

$$W = \mathrm{C}_\omega(g) = \big\{ h\, g\, \omega(h)^{-1}\ \big|\ h \in \mathrm{G} \big\} \tag{6.3}$$

of elements $g \in \mathrm{G}$ [2, 28, 83, 27, 14].

Definition 6.2. Two symmetric D-branes $\mathrm{C}_\omega(g)$ and $\mathrm{C}_\omega(g')$ are *equivalent* if $g' = \mathrm{Ad}_h(g) := h\, g\, h^{-1}$ for some $h \in \mathrm{G}$.

Equivalent D-branes are described by twisted conjugacy classes which are simply translates of one another in the group manifold of G. To characterize the corresponding equivalence classes, let Aut(G) denote the group of isometric automorphisms of G. Let $\mathrm{Inn(G)} \subset \mathrm{Aut(G)}$ be the invariant normal subgroup of inner automorphisms $g \mapsto \mathrm{Ad}_h(g)$. Then the factor group

$$\mathrm{Out(G)} = \mathrm{Aut(G)}\,/\,\mathrm{Inn(G)} \tag{6.4}$$

consists of equivalence classes of metric-preserving outer automorphisms of G. To each element of the group (6.4) we can associate an equivalence class of symmetric D-branes foliating G, because every element of G belongs to one and only one twisted conjugacy class. The leaves of this foliation need not all have the same topology. Hence the D-brane foliation need not be a fibration.

Thus a generic worldvolume (6.3) representing an equivalence class of D-branes is described by taking $\omega \in \mathrm{Out(G)}$ to be an outer automorphism. The symmetric D-brane is then called an *ω-twisted D-brane*. When $\omega = \mathrm{id}_\mathrm{G}$ then $\mathrm{C}(g) := \mathrm{C}_{\mathrm{id}_\mathrm{G}}(g)$ is just an ordinary *conjugacy class* of

the group G and the symmetric D-brane is called an *untwisted D-brane*. Generally, the twisted conjugacy class is diffeomorphic to a homogeneous space

$$\mathrm{C}_\omega(g) = \mathrm{G}\,/\,\mathrm{H}_\omega(g) \tag{6.5}$$

where

$$\mathrm{H}_\omega(g) = \big\{h \in \mathrm{G} \bigm| h\,g = g\,\omega(h)\big\} \tag{6.6}$$

is the isotropy subgroup of the element $g \in \mathrm{G}$.

Remark 6.1. It is possible to also construct D-branes with less symmetry, preserving a smaller subgroup than the diagonal $\mathrm{G} \subset \mathrm{G} \times \overline{\mathrm{G}}$. These are called *symmetry-breaking D-branes*. Generically, they are localized along *products* of twisted conjugacy classes of the Lie group G [73].

The supergravity fields on a symmetric D-brane are straightforward to construct [2, 83, 14]. For a given group element $g \in \mathrm{G}$, identify the tangent space $T_g\mathrm{G} \cong \mathfrak{g}$ to G at g with the Lie algebra of G. For tangent vectors $u, v \in \mathfrak{g}$ the invariant metric G is then given by

$$G(u,v) = \big\langle g^{-1}\,u\,,\,g^{-1}\,v\big\rangle\ , \tag{6.7}$$

while the invariant three-form H is

$$H(u,v,w) = -\big\langle g^{-1}\,u\,,\,\big[g^{-1}\,v\,,\,g^{-1}\,w\big]\big\rangle \tag{6.8}$$

for $u, v, w \in \mathfrak{g}$. We may introduce a B-field, with $H = \mathrm{d}B$, by the formula

$$B(u,v) = \Big\langle g^{-1}\,u\,,\,\frac{\mathbb{1} + \omega \circ \mathrm{Ad}_g}{\mathbb{1} - \omega \circ \mathrm{Ad}_g}\big(g^{-1}\,v\big)\Big\rangle \tag{6.9}$$

which is defined for $g^{-1}\,u \in \mathrm{im}(\mathbb{1} - \omega \circ \mathrm{Ad}_g)$, i.e. for vectors u tangent to the twisted conjugacy class (6.3) containing $g \in \mathrm{G}$. We assume that these fields restrict non-degenerately to the twisted conjugacy classes.

6.2 Untwisted D-branes

Let us first describe the noncommutative worldvolume geometries in the somewhat more standard cases with $\omega = \mathrm{id}_\mathrm{G}$. The D-brane worldvolumes are the conjugacy classes $W = \mathrm{C}(g) = \{\mathrm{Ad}_h(g) \mid h \in \mathrm{G}\}$ which are diffeomorphic to the symmetric spaces $\mathrm{G}\,/\,\mathrm{H}(g)$, where $\mathrm{H}(g)$ is the stabilizer subgroup of the element $g \in \mathrm{G}$. Since G is compact and simple, every element is conjugate to some maximal torus T of G. Let us restrict to the set G_r of *regular* elements $g \in \mathrm{G}$, i.e. those elements which are conjugate to only one maximal torus. The set G_r is an open dense subset in G. Let $\mathrm{T}_\mathrm{r} := \mathrm{T} \cap \mathrm{G}_\mathrm{r}$. Then $\mathrm{C}(g) \cap \mathrm{T}_\mathrm{r} \neq \emptyset$ and the intersections generate an orbit of the Weyl group $\mathrm{W} = \mathrm{N}(\mathrm{T}_\mathrm{r})\,/\,\mathrm{T}_\mathrm{r}$, where $\mathrm{N}(\mathrm{T}_\mathrm{r})$ is the normalizer subgroup of the maximal torus. Hence there is a diffeomorphism realizing the conjugacy class as the flag manifold

$$\mathrm{C}(g) = \mathrm{G}_\mathrm{r}\,/\,\mathrm{T}_\mathrm{r}\ \ . \tag{6.10}$$

Since the Weyl group W further relates elements of T_r, it follows that untwisted D-branes are parametrized by the Weyl chamber $\mathrm{T}_\mathrm{r}\,/\,\mathrm{W}$.

Our main result here is that the untwisted D-branes in G in the semi-classical regime recover the Kirillov theory of coadjoint orbits [3], whose quantization gives all irreducible representations of the universal enveloping algebra $U(\mathfrak{g})$.

Theorem 6.1. *To each irreducible representation $\mathcal{V}$ of the Lie group* G *there bijectively corresponds an untwisted D-brane in the semi-classical limit such that the noncommutative algebra of functions on the quantized worldvolume $W_{\mathcal{V}}$ is given by*

$$\mathcal{A}(W_{\mathcal{V}}) \;=\; \mathrm{End}(\mathcal{V}) \ .$$

Proof. The two-form B-field, defined on vectors tangent to the conjugacy class $\mathrm{C}(g)$, is given by the automorphism

$$B = \frac{\mathbb{1} + \mathrm{Ad}_g}{\mathbb{1} - \mathrm{Ad}_g} \ .$$

In the semi-classical limit, $H \to 0$ and the group manifold of G approaches flat space. We may then parametrize the conjugacy class by an element $X \in \mathfrak{g}$ such that $g \approx \mathbb{1} + X$ in the limit. The semi-classical B-field thereby becomes

$$B = -2\,(\mathrm{ad}_X)^{-1} \ ,$$

which is just the standard Kirillov two-form on the Lie algebra $\mathfrak{g}$. The Seiberg-Witten bivector is given by

$$\theta = \frac{2}{B - G\,B^{-1}\,G} = \tfrac{1}{2}\left(\mathrm{Ad}_{g^{-1}} - \mathrm{Ad}_g\right) \ .$$

In the limit $H \to 0$ this bivector obeys the Jacobi identity and becomes

$$\theta = \mathrm{ad}_X \ ,$$

which is the Kirillov-Kostant Poisson bivector. Casimir operators in $U(\mathfrak{g})$ are invariants of the conjugacy classes and hence may be used to label $\mathrm{C}(g)$. We may therefore associate to each conjugacy class $\mathrm{C}(g)$ whose second Casimir invariant is quantized in the requisite way an irreducible representation $\mathcal{V}_g$ of $U(\mathfrak{g})$. □

Remark 6.2. This theorem is consistent with the fact that conjugacy classes are the image under the exponential map of adjoint orbits, which are diffeomorphic to coadjoint orbits that are symplectic manifolds with respect to the natural Kirillov-Kostant-Souariu symplectic structure and hence are even-dimensional. Note that the invariant inner product $\langle -, - \rangle$ on $\mathfrak{g}$ plays a crucial role in this identification. It relates the D-brane worldvolume, which is an orbit of the *adjoint* action of the group G on itself, to the *coadjoint* action of G on its Lie algebra $\mathfrak{g}$. The quantization of the coadjoint orbits in turn gives the representations of G.

Remark 6.3. The proof of Theorem 6.1 shows that the semi-classical geometry of untwisted D-branes is very close to that of the cases studied in earlier sections with constant B-field [3]. Since G is compact, the algebra $\mathcal{A}(W_{\mathcal{V}})$ is finite-dimensional and so the noncommutative worldvolume geometry is now "fuzzy". This algebra carries a natural G-action on it which represents the isometry group of the noncommutative space. Note that not all conjugacy classes are admissible as D-brane worldvolumes. Only the *integer* conjugacy classes which are in one-to-one correspondence with the irreducible representations of G are allowed. The algebra of functions on the quantized conjugacy classes is then the corresponding endomorphism algebra of the representation.

Remark 6.4. In some instances Theorem 6.1 also extends to the case of non-compact and even non-semisimple Lie groups G, such as those corresponding to homogeneous plane wave backgrounds [42]. It is not clear how to generalize this result to symmetry-breaking D-branes whose classical worldvolumes are given as products of conjugacy classes [73].

6.3 Example: D-branes in SU(2)

Let us now study in detail the simplest example of this construction [3, 4, 79]. The rank 1 Lie group $\mathrm{G} = \mathrm{SU}(2)$ has no non-trivial Dynkin diagram automorphisms and hence all symmetric D-branes in $\mathrm{SU}(2) \cong \mathrm{S}^3$ are untwisted. In the semi-classical limit $H \to 0$, the radius of the three-sphere S^3 grows and the group manifold "decompactifies" onto flat space $\mathbb{R}^3 = \mathrm{su}(2)$. The conjugacy classes are parametrized by the Weyl chamber $\mathrm{S}^1 \,/\, \mathbb{Z}_2$. Let $\vartheta \in [0,\pi]$ be the coordinate on this closed interval. Then ϑ parametrizes the azimuthal angle of S^3. For $\vartheta = 0, \pi$, the conjugacy classes are points corresponding to elements in the center $\mathbb{Z}_2 = \{-\mathbb{1}, \mathbb{1}\}$ of SU(2) placed at the north and south poles of the three-sphere. They correspond to D0-branes. For $\vartheta \in (0,\pi)$ the conjugacy classes are diffeomorphic to two-spheres $\mathrm{S}^2 \cong \mathrm{S}^3 \,/\, \mathrm{S}^1$, corresponding to D2-branes. The group manifold $\mathrm{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ has a standard foliation by two-spheres of increasing radius with two degenerate spheres S^2 placed at 0 and ∞. Because of the degeneration of the limiting spheres, the foliation is not a fibration (In particular this is *not* the Hopf fibration $\mathrm{S}^3 \to \mathrm{S}^2$ that we are describing here).

In this case $\theta = \mathrm{ad}_X$ is the standard Kirillov-Kostant Poisson bivector on the two-spheres in $\mathrm{su}(2) = \mathbb{R}^3$, and the worldvolume algebras of D-branes in SU(2) are the usual quantizations of these two-spheres via the coadjoint orbit method. By Theorem 6.1, quantizing functions on S^2 with the usual Poisson structure yields *fuzzy spheres* [61]

$$\mathcal{A}\big(\mathrm{S}^2_j\big) = \mathbb{M}_N(\mathbb{C}) \tag{6.11}$$

which are labelled by half-integers $j \in \frac{1}{2}\,\mathbb{N}_0$. The D-brane label j is proportional to the radius of its worldvolume S^2 and it represents the spin of the associated irreducible SU(2)-module $\mathcal{V}_j$ of dimension $N = 2j+1$. The radii R_j of the corresponding integer conjugacy classes in SU(2) are quantized as $R_j^2 = j\,(j+1)$ in order to match the corresponding second Casimir invariants.

The worldvolume algebra (6.11) has finite dimension $(2j+1)^2$. It is thus a full matrix algebra which admits an action of SU(2) by conjugation with group elements in the N-dimensional representation of SU(2). Under this action, the SU(2)-module $\mathbb{M}_N(\mathbb{C})$ decomposes into a direct sum of irreducible representations $\mathcal{V}_J$ of dimension $2J+1$, giving [61]

$$\mathcal{A}\big(\mathrm{S}^2_j\big) = \bigoplus_{J=0}^{N-1} \mathcal{V}_J \ . \tag{6.12}$$

Note that only integer values of the spin J appear in this direct sum. Let Y_a^J, $a = 1, \dots, 2J+1$ be a basis of the representation space $\mathcal{V}_J$. These elements are called *fuzzy spherical harmonics* and their multiplication rules can be worked out from the multiplication of $N \times N$ matrices to get [61]

$$Y_a^I\,Y_b^J = \sum_{K=0}^{\min(I+J,2j)} ~ \sum_{c=1}^{2K+1} \left[\begin{smallmatrix} I & J & K \\ a & b & c \end{smallmatrix}\right] \left\{\begin{smallmatrix} I & J & K \\ j & j & j \end{smallmatrix}\right\} \, Y_c^K \ , \tag{6.13}$$

where the square brackets denote the Clebsch-Gordan coefficients of SU(2) and the curly brackets are the Wigner $6j$-symbols of $U(\mathrm{su}(2))$.

To construct field theories on the noncommutative worldvolumes (6.11), we will introduce derivations on $\mathcal{A}(\mathrm{S}^2_j)$ in analogy with the flat space case in Section 2.4 by finding automorphisms of the noncommutative algebra which represent isometries of the sphere S^2. Recall that the classical su(2) symmetry on the commutative algebra of functions $C(\mathrm{S}^2)$ is inherited from infinitesimal rotations in $\mathbb{R}^3$ as follows. Let $y=(y^i)_{i=1,2,3}$ denote local coordinates on $\mathbb{R}^3$. Let C^{ijk} be the structure constants of the Lie algebra su(2) in a suitable basis. Then

$$L_i=\sum_{j,k=1}^3 C^{ijk}\,y^j\,\frac{\partial}{\partial y^k}~, \quad i=1,2,3 \tag{6.14}$$

generate an su(2) subalgebra in the Lie algebra of vector fields on $\mathbb{R}^3$. The classical functions Y^J_a above are then the usual spherical harmonics transforming under L_i in the representation $\mathcal{V}_J$ of su(2). Under the embedding $\mathrm{S}^2\subset\mathbb{R}^3$ defined by $|y|^2=1$, this descends to an action of SU(2) on the algebra $C(\mathrm{S}^2)$.

Let us now write down the quantization of this su(2) symmetry. Let $f=\sum_{J,a} f_{Ja}\,Y^J_a$ be an element of the worldvolume algebra (6.12). In analogy with the flat space case, we then introduce derivatives through the adjoint representation of su(2) by

$$L_i(f):=\big[\mathrm{Y}^1_i\,,\,f\big]~, \quad i=1,2,3~. \tag{6.15}$$

The automorphisms $L_i:\mathcal{A}(\mathrm{S}^2_j)\to\mathcal{A}(\mathrm{S}^2_j)$ generate the reducible action of SU(2) on the noncommutative worldvolume algebra under which it decomposes as in eq. (6.12).

We are now ready to describe gauge theory on the noncommutative space (6.11). As always, the standard procedure for constructing connections and gauge theories in noncommutative geometry can be formally developed in this instance [61]. Here we will just write down the final result for a rank 1 gauge theory on the trivial projective module over the fuzzy sphere. We express an arbitrary connection $\nabla_i:\mathcal{A}(\mathrm{S}^2_j)\to\mathcal{A}(\mathrm{S}^2_j)$ in this case in the form

$$\nabla_i=L_i+A_i \tag{6.16}$$

with $A_i\in\mathrm{i}\,\mathrm{u}(N)$, $i=1,2,3$. Unlike the previous flat space actions, string theory considerations [4] dictate that the gauge theory action functional $S:\,\mathrm{i}\,\mathrm{u}(N)\to\mathbb{R}$ contains both Yang-Mills and Chern-Simons terms in the combination

$$S(A)=\ \mathrm{Tr}\,\Big(\sum_{i,j=1}^3\Big[\big(F_{ij}\big)^2+2\sum_{k=1}^3 C^{ijk}\ \mathrm{CS}_{ijk}(A)\Big]\Big)~. \tag{6.17}$$

Here Tr denotes the usual $N\times N$ matrix trace on $\mathbb{M}_N(\mathbb{C})$, and

$$F_{ij}:=\big[\nabla_i\,,\,\nabla_j\big]=L_i(A_j)-L_j(A_i)+\big[A_i\,,\,A_j\big]-\mathrm{i}\sum_{k=1}^3 C^{ijk}\,A_k \tag{6.18}$$

is the curvature of the connection (6.16). The functional

$$\mathrm{CS}_{ijk}(A)=\ \mathrm{i}\,L_i(A_j)\,A_k+\tfrac13\,A_i\,\big[A_j\,,\,A_k\big]+\tfrac12\sum_{l=1}^3 C^{ijl}\,A_l\,A_k \tag{6.19}$$

is the *noncommutative Chern-Simons form* [4]. The action functional (6.17) is invariant under the (infinitesimal) gauge transformations

$$A_i \longmapsto A_i + L_i(\Lambda) - \mathrm{i}\,\big[A_i\,,\,\Lambda\big] \tag{6.20}$$

with $\Lambda \in \mathrm{u}(N)$.

Varying the functional (6.17) gives the equations of motion

$$\sum_{i=1}^{3}\,\big(L_i(F_{ij}) + \big[A_i\,,\,F_{ij}\big]\big) = 0\ , \quad j = 1,2,3 \tag{6.21}$$

expressing the usual fact that the curvature is covariantly constant. To solve these equations, introduce the shifted algebra elements

$$R_i := Y_i^1 + A_i \tag{6.22}$$

in $\mathbb{M}_N(\mathbb{C})$ to write them as

$$\sum_{i=1}^{3}\big[R_i\,,\,\big[R_i\,,\,R_j\big] - \sum_{k=1}^{3} C^{ijk}\,R_k\big] = 0\ . \tag{6.23}$$

There are then two classes of solutions. The first class arises from requiring that all three matrices (6.22) be mutually commuting in $\mathbb{M}_N(\mathbb{C})$, $[R_i, R_j] = 0$ for $i,j = 1,2,3$. They can therefore be simultaneously diagonalized and their simultaneous eigenvalues describe translates of N particles in the group target space $X = \mathrm{SU}(2)$. These are formally the same as the solutions found earlier in the cases of flat spaces [89].

A more interesting class of solutions with no flat space analog is provided by those configurations (6.22) which obey the commutation relations

$$\big[R_i\,,\,R_j\big] = \sum_{k=1}^{3} C^{ijk}\,R_k\ . \tag{6.24}$$

Such solutions have vanishing curvature $F_{ij} = 0$ and thus correspond to flat connections on the D-brane worldvolume. They determine N-dimensional unitary representations of su(2), i.e. homomorphisms

$$\pi_N\ :\ \mathrm{su}(2) \longrightarrow \mathrm{u}(N)\ . \tag{6.25}$$

Up to isomorphism, for any $n \in \mathbb{N}$ there is a unique irreducible representation of SU(2) of dimension n. Thus to any representation (6.25) we can assign an unordered *partition* $(n_i)_{i=1,\dots,r}$ of $N = n_1 + \dots + n_r$, with n_i giving the dimensions of the irreducible submodules in π_N. These configurations are thus similar to the torus instantons that we constructed in Section 5.3. The partition characterizes the original representation uniquely up to gauge equivalence, and hence provides a simple classification for solutions of this type. It is also possible to formulate gauge theory on the fuzzy sphere in such a way that the classical solution set resembles more closely that of Yang-Mills theory on $\mathbb{V}_2^\theta$ and T_Θ^2, with intimate relationships between the seemingly distinct instanton configurations on the diverse noncommutative spaces [84, 85].

Remark 6.5. In the string theory setting, these solutions describe dynamical processes involving a stack of N particles corresponding to D0-branes, which are labelled by quantum mechanical instanton-type partition degrees of freedom. These D0-branes "decay" into a single D2-brane (for rank 1 gauge theory) with spherical worldvolume S^2 corresponding to the irreducible representation $\mathcal{V}_j$ of dimension $N = 2j+1$ [4]. This condensation phenomenon is called the *dielectric effect* [65] and it is equivalent to vector bundle modification when D-branes are regarded as Baum-Douglas K-cycles in topological K-homology [46, 8, 74, 87].

6.4 Twisted D-branes

We close this final section with a tour beyond the SU(2) target space and untwisted D-branes. Let us consider the generic case of an ω-twisted D-brane corresponding to a non-trivial outer automorphism ω of the Lie group G. In this case, the dimension of a twisted conjugacy class $\mathrm{C}_\omega(g)$ is larger than the dimension of a regular conjugacy class. In particular, there are generically instances in which $\mathrm{C}_\omega(g)$ is odd-dimensional, and so in general the worldvolume W will not be a symplectic manifold. Nevertheless, there is a way to quantize these geometries that we shall now describe.

The main result here is that the ω-twisted D-branes in G are labelled by representations of the *invariant subgroup*

$$\mathrm{G}^\omega := \big\{ g \in \mathrm{G} \ \big| \ \omega(g) = g \big\} \tag{6.26}$$

which for $\omega \neq \mathrm{id}_{\mathrm{G}}$ is a proper subgroup of G. By conjugating to the maximal torus T, they are thus parametrized by the abelian subgroup $\mathrm{T}^\omega := \mathrm{G}^\omega \cap \mathrm{T}$ and there is a diffeomorphism

$$\mathrm{C}_\omega(g) = \mathrm{G} \, / \, \mathrm{T}^\omega \tag{6.27}$$

for any $g \in \mathrm{G}$. In the semi-classical regime, the quantization of twisted conjugacy classes, i.e. the noncommutative geometry of twisted D-branes, can be described as follows [5].

Theorem 6.2. *To each irreducible representation $\mathcal{V}^\omega$ of the invariant subgroup G^ω there bijectively corresponds an ω-twisted D-brane in the semi-classical limit such that the noncommutative algebra of functions on the quantized worldvolume $W_{\mathcal{V}^\omega}$ is given by*

$$\mathcal{A}\big(W_{\mathcal{V}^\omega}\big) \ = \ \big(C(\mathrm{G}) \otimes \mathrm{End}(\mathcal{V}^\omega)\big)^{\mathrm{G}^\omega} \ ,$$

where the superscript denotes the G^ω-invariant part and the G^ω-action $\mathrm{G}^\omega \times C(\mathrm{G}, \mathrm{End}\, \mathcal{V}^\omega) \to C(\mathrm{G}, \mathrm{End}\, \mathcal{V}^\omega)$ is defined by $(h, f(g)) \mapsto \mathcal{V}^\omega(h) \, f(g\,h) \, \mathcal{V}^\omega(h)^{-1}$ with $\mathcal{V}^\omega(h) \in \mathrm{GL}(\mathcal{V}^\omega)$ for all $h \in \mathrm{G}^\omega$.

Proof. Consider an open neighbourhood U of the identity element $\mathbb{1}$ of G. Let $g \in U$. Then the twisted conjugacy class of g can be represented as the fibration

$$\mathrm{C}_\omega(g) = \mathrm{G} \times_{\mathrm{G}^\omega} \mathrm{C}'(g)$$

over $\mathrm{G} \, / \, \mathrm{G}^\omega$ with fiber $\mathrm{C}'(g)$ which is a regular conjugacy class of the invariant subgroup G^ω. Here G^ω acts on G by right multiplication. As before in the untwisted case, in the semi-classical limit $H \to 0$ the conjugacy class $\mathrm{C}'(g)$ becomes small and approaches a coadjoint orbit of G^ω,

while the Poisson manifold $\mathrm{G}\,/\,\mathrm{G}^\omega$ grows (approaching flat space) and its Poisson bivector scales down. Thus $\mathrm{C}'(g)$ becomes a noncommutative symplectic space while $\mathrm{G}\,/\,\mathrm{G}^\omega$ remains a classical space in the semi-classical regime. After quantization, we get a bundle with noncommutative fibers $\mathrm{End}(\mathcal{V}_g^\omega)$ and a classical base $\mathrm{G}\,/\,\mathrm{G}^\omega$. □

Remark 6.6. $\mathcal{A}(W_{\mathcal{V}^\omega})$ is an associative matrix algebra of functions on the Lie group G. If $\mathcal{V}^\omega \cong \mathbb{C}$ is the trivial representation of G^ω, then the noncommutative algebra $\mathcal{A}(W_{\mathbb{C}})$ consists of functions on G which are simply invariant under right translations by elements of the invariant subgroup $\mathrm{G}^\omega \subset \mathrm{G}$. On the other hand, when $\omega = \mathrm{id}_{\mathrm{G}}$ is the trivial automorphism of G, one has $\mathrm{G}^\omega = \mathrm{G}$ and the noncommutative worldvolume algebra is $\mathrm{End}(\mathcal{V}^{\mathrm{id}_{\mathrm{G}}})$ consistently with Theorem 6.1.

Using the fact that the Lie group G is simple, simply-connected and compact, we can obtain an alternative realization of the noncommutative worldvolume algebra $\mathcal{A}(W_{\mathcal{V}^\omega})$ which makes its G-module structure more transparent.

Theorem 6.3. *All complexified vector bundles* $\mathrm{G} \times_{\mathrm{G}^\omega} (\mathcal{V}^\omega)^{\mathbb{C}} \longrightarrow \mathrm{G}\,/\,\mathrm{G}^\omega$ *are trivial.*

The proof of Theorem 6.3 is rather technical and can be found in [5].

Proposition 6.1. *There is a natural algebra isomorphism*

$$\mathcal{A}\big(W_{\mathcal{V}^\omega}\big) \ \cong \ \big\{f \in C\big(\mathrm{G}\,,\,\mathrm{End}\,\mathcal{V}^\omega\big) \ \big| \ f(g\,h) = \mathcal{V}^\omega(h)^{-1}\,f(g)\,\mathcal{V}^\omega(h)\ , \ g \in \mathrm{G}\,,\,h \in \mathrm{G}^\omega\big\}\ .$$

Proof. The vector space $C(\mathrm{G}) \otimes \mathrm{End}(\mathcal{V}^\omega) \cong C(\mathrm{G}, \mathrm{End}\,\mathcal{V}^\omega)$ of matrix-valued functions on G carries a natural $(\mathrm{G} \times \mathrm{G}^\omega)$-action $(\mathrm{G} \times \mathrm{G}^\omega) \times C(\mathrm{G}, \mathrm{End}\,\mathcal{V}^\omega) \to C(\mathrm{G}, \mathrm{End}\,\mathcal{V}^\omega)$ given by

$$\big((g,h)\,,\,f(g')\big) \ \longmapsto \ \mathcal{V}^\omega(h)\,f\big(g^{-1}\,g'\,h\big)\,\mathcal{V}^\omega(h)^{-1}\ .$$

This leaves an action of G on the space $\mathcal{A}(W_{\mathcal{V}^\omega})$ of G^ω-invariants. The G-module $\mathcal{A}(W_{\mathcal{V}^\omega})$ can thereby be realized explicitly in terms of G^ω-equivariant functions on G as claimed. □

To construct gauge theory on the trivial rank 1 module over the algebra $\mathcal{A}(W_{\mathcal{V}^\omega})$, let T^a, $a = 1, \dots, \dim(\mathrm{G})$ be a basis of generators of the Lie algebra $\mathfrak{g}$ obeying

$$\big[T^a\,,\,T^b\big] = \sum_{c=1}^{\dim(\mathrm{G})} C^{abc}\,T^c\ . \tag{6.28}$$

Using the G-action on the algebra $\mathcal{A}(W_{\mathcal{V}^\omega})$ given by Proposition 6.1 and the exponential mapping $\exp : \mathfrak{g} \to \mathrm{G}$, we define Lie derivatives $L_a(f)$ of functions $f \in \mathcal{A}(W_{\mathcal{V}^\omega})$ by

$$\big(L_a(f)\big)(g) := \frac{\mathrm{d}}{\mathrm{d}t} f\big(\exp(-t\,T^a)\,g\big)\Big|_{t=0}\ , \qquad a = 1, \dots, \dim(\mathrm{G}) \tag{6.29}$$

which as vector fields on $\mathfrak{g}$ obey the same Lie algebra relations (6.28). As in Section 6.3, the natural string-inspired gauge theory action functional $S_\omega : \mathcal{A}(W_{\mathcal{V}^\omega}) \to \mathbb{R}$ reads [5]

$$S_\omega(A) = \int_{\mathrm{G}} \mathrm{d}\mu_{\mathrm{G}} \ \sum_{a,b=1}^{\dim(\mathrm{G})} \ \mathrm{Tr}\,\Big(\big(F_{ab}\big)^2 + 2 \sum_{c=1}^{\dim(\mathrm{G})} C^{abc}\ \mathrm{CS}_{abc}(A)\Big) \tag{6.30}$$

with all objects defined in a completely analogous way to those of Section 6.3. Here $\mathrm{d}\mu_{\mathrm{G}}$ is the invariant Haar measure on the Lie group G. The classical solutions of this gauge theory again describe condensation processes on a configuration of D-branes which drive the entire system into another D-brane configuration [5].

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